DEFORMATION RINGS AND HECKE ALGEBRAS IN THE TOTALLY REAL CASE

KAZUHIRO FUJIWARA

Abstract. In this paper, the ℓ-adic Hecke algebras of GL_2 over totally real fields are studied. In particular we show that they are identified with deformation rings of mod ℓ Galois representations in important cases, assuming that the representation is absolutely irreducible.

Contents

0. Introduction

One of the basic questions in number theory is to determine semi-simple ℓ-adic representations of the absolute Galois group of a number field F. In case of abelian representations, a satisfactory answer is known for compatible system of ℓ-adic representations, and these types of abelian representations are obtained from algebraic Hecke characters. In this paper, we discuss the question for two dimensional representations over a totally real number field.

For a totally real field F, let $G_F = \mathrm{Gal}(\bar{F}/F)$ be the absolute Galois group, and $I_{F,\infty} = \{\iota : F \hookrightarrow \mathbb{R}\}$ be the set of the infinite places of F. We take a pair (k, w) of integral vector $k = (k_\iota)_{\iota \in I_{F,\infty}} \in \mathbb{Z}^{I_{F,\infty}}$ and an integer $w \in \mathbb{Z}$ called discrete infinity type (cf. Definition 4.1). For a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$, we assume that the infinity part π_∞ is isomorphic to $\bigotimes_{\iota \in I_{F,\infty}} D_{k\iota,w}$. Here $D_{k_\iota,w}$ is an essentially square integrable representation of $\mathrm{GL}_2(\mathbb{R})$ (see Conventions for our normalization). These types of cuspidal representations are generated by holomorphic Hilbert modular forms.

Fix a prime ℓ, and an isomorphism $\mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$ by the axiom of choice. It is known that the finite part π_f of π as above is defined over some ℓ-adic field E_λ with the integer ring o_{E_λ} and the residue field k_λ, and there is a two dimensional continuous λ-adic Galois representation

$$\rho_{\pi,E_\lambda} : G_F \to \mathrm{GL}_2(o_{E_\lambda})$$

associated to π, which is pure of weight $w+1$ with respect to geometric Frobenius elements, and $\det\rho_{\pi,E_\lambda} \cdot \chi_{\mathrm{cycle}}^{w+1}$ is a character of finite order (see [33], [6], [50], [46], [3] for $F \neq \mathbb{Q}$). Here $\chi_{\mathrm{cycle}} : G_F \to \mathbb{Z}_\ell^\times$ is the cyclotomic character. We call this class of λ-adic representations *modular* λ-adic representations. Conjecturally, modular λ-adic representations in our

sense should cover most *motivic* and *totally odd* Galois representations (global Langlands correspondence). When the ℓ-adic representation ρ is obtained from an elliptic curve E over F, this conjecture is a generalized form of the Taniyama-Shimura conjecture.

In [51], in case of $\mathbb{Q}$, Wiles has studied the problem of modularity via the deformation theory of mod ℓ-Galois representations. We take the same approach in this paper. For a mod ℓ-representation $\bar{\rho} : G_F \to \mathrm{GL}_2(k_\lambda)$ having values in k_λ, assume that there is a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of infinity type (k, w), and $\bar{\rho}$ is obtained as the reduction modulo λ of the associated Galois representation, that is,

$$\bar{\rho} \simeq \rho_{\pi, E_\lambda} \mod \lambda.$$

We call $\bar{\rho}$ *modular* if this condition is satisfied.

Instead of the global Langlands correspondence itself, we consider the following question, which asks the stability of the notion of the modularity of λ-adic representations under perturbation:

Question 0.1. *Assume that mod ℓ-representation $\bar{\rho}$ is modular, and take a λ-adic representation ρ which lifts $\bar{\rho}$. Is ρ modular?*

Without any restriction, ρ can not be modular, since ρ_{π, E_λ} is expected to be *motivic*, which is known in most cases ([6], [3]). Thus there are strong restrictions on local monodromies of ρ, in particular at the places dividing ℓ.

On the other hand, ρ is seen as an ℓ-adic *deformation* of $\bar{\rho}$ from the viewpoint of Mazur if $\bar{\rho}$ is irreducible [32]. So one may expect that if the local conditions are imposed appropriately, all reasonable λ-adic deformations of $\bar{\rho}$ are modular representations. Hida introduced ordinary ℓ-adic Hecke algebras and constructed Galois representations having values in it when $F = \mathbb{Q}$ [22]. Mazur conjectured these Hecke algebras introduced by Hida are actually the universal deformation rings which control all deformations with suitable local monodromy conditions.

In [51], together with [49], it was shown affirmatively that the ℓ-adic Hecke algebra is the universal deformation ring in the case where $F = \mathbb{Q}$, including some non-ordinary cases (see [9] for a generalization). Moreover, Wiles applied this result to the modularity of λ-adic representations, and proved the Taniyama-Shimura conjecture over $\mathbb{Q}$ in the semistable case.

Our main theorem in this article is the following (Theorem 11.1):

Theorem 0.2 ($R = T$ theorem)**.** *Let F be a totally real number field of degree d, $\bar{\rho} : G_F \to \mathrm{GL}_2(k)$ an absolutely irreducible mod ℓ-representation. We fix a deformation type $\mathscr{D}$, and assume the following conditions.*

(1) *$\ell \geq 3$, and $\bar{\rho}|_{F(\zeta_\ell)}$ is absolutely irreducible. When $\ell = 5$, the following case is excluded: the projective image $\bar{G}$ of $\bar{\rho}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, and the mod ℓ-cyclotomic character $\bar{\chi}_{\mathrm{cycle}}$ factors through $G_F \to \bar{G}^{\mathrm{ab}} \simeq \mathbb{Z}/2$ (in particular $[F(\zeta_5) : F] = 2$).*

(2) *For $v|\ell$, the deformation condition for $\bar{\rho}|_{G_{F_v}}$ is either nearly ordinary or flat. When the condition is nearly ordinary (resp. flat) at v, we assume that $\bar{\rho}|_{G_{F_v}}$ is G_{F_v}-distinguished (resp. F_v is absolutely unramified).*

(3) *There is a minimal modular lifting π of $\bar{\rho}$ in Definition 6.11.*

(4) *Hypothesis 6.7 is satisfied.*

(5) *If $\mathscr{D}$ is not minimal, we assume Hypothesis 5.9 when $q_{\bar{\rho}} = 1$.*

Then the universal deformation ring $R_{\mathscr{D}}$ of $\bar{\rho}$ of type $\mathscr{D}$ is a complete intersection, and is isomorphic to the Hecke algebra $T_{\mathscr{D}}$.

As is already remarked, this is proved in [49], [51], [9] in the case where $F = \mathbb{Q}$ (our method in this paper gives a substantial simplification, which is also due to Diamond [10]). Theorem 11.1 is a basic tool to study the modularity questions. As in [51], one deduces the modularity of some 3 and 5-adic representations from the theorem.

As another application of Theorem 11.1, we obtain the finiteness of the Selmer group for the adjoint representation.

Corollary 0.3. *Under the same assumption as in Theorem 11.1, the Selmer group* $\mathrm{Sel}_{\mathscr{D}}(F,\ \mathrm{ad}\,\rho)$ *of* $\rho = \rho_{\pi,E_\lambda}$ *for* π *appearing in* $T_{\mathscr{D}}$ *is finite.*

For other applications of the main theorem, see [16].

Here is the explanation of the conditions of the main theorem.

In (1), the exceptional case when $\ell = 5$ does not happen in the application to elliptic curves (see Proposition 9.8). Even in the exceptional case, Theorem 11.1 holds true by a slight generalization of our method, which will be discussed on another occasion.

In (2), F_v can be absolutely ramified if the deformation condition is nearly ordinary at v.

For (3), if one only assumes that $\bar{\rho}$ is modular, the existence of a minimal lift is satisfied outside ℓ by [25], [26], [15], and [35]. At places dividing ℓ, the existence is satisfied if the condition is nearly ordinary. In general, only a partial answer is known.

In (4), Hypothesis 6.7 (local monodromy hypothesis) is satisfied if $k = (2, \ldots, 2)$, or the deformation condition is nearly ordinary at all $v|\ell$. So the theorem is applied to Hida's nearly ordinary Hecke algebras [24].

As for (5), Hypothesis 5.9 is what is called as "Ihara's Lemma for Shimura curves". Ribet [38] proved it when $F = \mathbb{Q}$ and modular curves using a result of Ihara. If one is only interested in the modularity question, Hypothesis 5.9 can be avoided by choosing a totally real quadratic extension F' of F which depends on $\bar{\rho}$, and using a base change argument. We will discuss the local monodromy hypothesis and Ihara's Lemma for Shimura curves on another occasion.

Our proof of Theorem 11.1 consists of two basic steps, similarly to the argument in [51]. One first proves the theorem when the deformation condition $\mathscr{D}$ is minimal, and reduces the general case to the minimal case by a level raising argument.

In the minimal case, the problem of showing that $R_{\mathscr{D}}$ is equal to $T_{\mathscr{D}}$ ($R = T$ theorem, a generalized form of Mazur's conjecture) is reduced to the construction of a family of rings and modules $\{R_Q, M_Q\}_{Q \in X}$ with a weak compatibility, which we call a *Taylor-Wiles system.* Here an element $Q \in X$ of the index set X is a finite set of finite places, and $\emptyset \in X$. $R_\emptyset = R_{\mathscr{D}}$, R_Q is a complete noetherian local o_{E_λ}-algebra with a surjective map $R_Q \twoheadrightarrow R_{\mathscr{D}}$, and M_Q is an R_Q-module. The axioms of Taylor-Wiles systems and the fundamental theorem (Theorem 2.3) called the *complete intersection-freeness criterion* are discussed in §2. $R = T$ theorem in the minimal case is proved by applying this criterion. A similar observation was also made independently by Diamond [10]. This is one of the main innovations which made possible our generalization to totally real fields. In [49], Taylor-Wiles system $\{R_Q, M_Q\}_{Q \in X}$ was constructed for Hecke algebras in case of $\mathbb{Q}$. There R_Q is a certain Hecke algebra T_Q dominating $T_{\mathscr{D}}$, and M_Q is known to be free over R_Q by an application of the q-expansion principle (the freeness theorem). The freeness of M_Q over T_Q (which implies the Gorensteinness of T_Q in the modular curve case) goes back to Mazur's work. Though the q-expansion principle is also available for Hilbert modular forms, a direct application of the method used for modular curves seems more difficult, and to show the freeness theorem and the Gorensteinness of the Hecke algebras in the totally real case, Taylor-Wiles systems are the most powerful tools at present.

In our formulation, notice that M_Q is not assumed to be free over R_Q, but is finite free over some *group ring* $o_{E_\lambda}[\Delta_Q]$. The freeness over a group ring plays an essential role in showing that a well-chosen limit of R_Q is a power series ring, and behaves as if R_Q were smooth rings.

In our general totally real case, a Taylor-Wiles system is constructed for universal deformation rings. A direct use of the universal deformation rings is suggested by an observation of Faltings ([49], appendix). M_Q is constructed from the middle dimensional cohomology group of some modular variety S_Q attached to a quaternion algebra D over F of complex dimension ≤ 1. S_Q is a Shimura curve, or a zero-dimensional variety associated to definite quaternion algebra whose arithmetic importance was found by Hida [23]. In the latter case, S_Q is *not* a Shimura variety in the sense of Deligne, which we call a *Hida variety.* To verify the axiom TW4, we make use of an argument based on a property of perfect complexes. The perfect complex argument is extremely useful in the study of congruences. For example, if one can control the mod ℓ-cohomology groups except the middle dimension, one may expect that the argument given in the paper is effective for other modular varieties, and Taylor-Wiles systems are obtained from the middle dimensional cohomology groups.

After the construction of a Taylor-Wiles system, we deduce $R = T$ theorem and the complete intersection property of the Hecke algebra in the minimal case at the same time by the complete intersection-freeness criterion. In particular we prove that the minimal Hecke algebra is a complete intersection without knowing a priori the freeness nor the Gorenstein property. Moreover, the freeness of the middle dimensional cohomology also follows *as a consequence* (Theorem 11.1).

Theorem 0.4 (Freeness theorem)**.** *Under the same assumption as in $R = T$ theorem, $M_\phi = M_{\mathscr{D}}$ is a free $T_{\mathscr{D}}$-module.*

For general deformation type $\mathscr{D}$, the three properties

- $R = T$,
- the local complete intersection property of $T_{\mathscr{D}}$,
- the freeness theorem

are proved inductively from the minimal case *at the same time* by calculating cohomological congruence modules in the sense of Ribet [38]. Hypothesis 5.9 is needed in this calculation. The level raising formalism is given in §2 (one may also use results of Diamond [10]). Since we do not know the Gorenstein property of $T_{\mathscr{D}}$ in advance, to deduce $R = T$, we use Lenstra's criterion [31] instead of Wiles' criterion [51]. The argument of the freeness given in §2 is due to T. Saito.

Throughout this paper, we make a systematic use of the homological algebra on Shimura varieties. Some foundational results are treated as general as possible for the applications in future including absolutely reducible representations.

This paper is organized as follows.

In §2, a formulation and a generalization of a beautiful argument in [49] is given. The notion of Taylor-Wiles systems is introduced. Our formulation is influenced by an argument of Faltings ([49], appendix). As is remarked already, a similar observation was made independently by Diamond [10]. We also give a formulation of the level raising argument.

In §3, we discuss deformations of local and global Galois deformations. We include the case treated by Diamond [9], though we make some modifications using Géradin's result [19].

In §4, basic results on modular varieties are discussed. In particular we treat general coefficient sheaves. The duality formalism is needed in the later sections.

In §5, the universal exactness of cohomological sequences is formulated and studied. The universal exactness (Ihara's Lemma) is proved for definite quaternion algebras except a class of absolutely reducible representations which is called of residual type.

In §6, nearly ordinary automorphic representations are discussed. We formulate a property on local monodromies of Galois representations attached to such representations as Local monodromy hypothesis (Hypothesis 6.7). The hypothesis is known to hold in many cases.

In §7, we construct and study a Hecke algebra $T_{\mathscr{D}}$ and a Galois representation $\rho_{\mathscr{D}}^{\mathrm{mod}}$ having values in $\mathrm{GL}_2(T_{\mathscr{D}})$ for deformation type $\mathscr{D}$ of $\bar{\rho}$ using the irreducibility of $\bar{\rho}$. For the construction of $\rho_{\mathscr{D}}^{\mathrm{mod}}$, one applies the method of pseudo-representation of Wiles, and reduces it to the existence of λ-adic representation ρ_{π,E_λ} attached to a cuspidal representation π. $\rho_{\mathscr{D}}^{\mathrm{mod}}$ is seen as the universal modular deformation of $\bar{\rho}$. To have a ring homomorphism $\pi_{\mathscr{D}} : R_{\mathscr{D}} \to T_{\mathscr{D}}$ from the universal deformation ring, one needs to check the local property of $\rho_{\mathscr{D}}^{\mathrm{mod}}$ at all places. The necessary information at $v \nmid \ell$ follows from the following fact: the global Langlands correspondence $\pi \mapsto \rho_{\pi,E_\lambda}$ is compatible with the local Langlands correspondence [29]. This compatibility is shown in [6], [51], [46]. For places dividing ℓ, the situation is much subtle. If the deformation condition is flat, we prove it by an arithmetic geometric analysis of Shimura curves using [5] if the quaternion algebra D is indefinite. If D is definite, one can not treat it directly, and we use the approximation method of Taylor [46] , [47]. The claim is reduced to the case of Shimura curves associated with some indefinite and ramified quaternion algebras.

In §8, we construct a Taylor-Wiles system from universal deformation rings and cohomology of modular varieties of complex dimension ≤ 1 as is remarked already. We do not use arithmetic geometrical properties of these "modular varieties".

In §9, we combine the results obtained in the previous sections, and deduce the main theorem in the minimal case. The deformation rings are controlled by Galois cohomology groups.

In §10, the calculation of cohomological congruence modules is done as in [51], §2.

In §11, we summarize the previous results, and the main theorem is proved.

Acknowledgement:

The first version of this paper was written during the author's stay at Münster university in 1996. There, only the minimal case was treated. The author would like to thank P. Schneider for hospitality. The second version was written during the author's stay at the Institute for Advanced Study for the academic year 1998-1999, which included the non-minimal cases. The present paper is based on the second version. The author would like to thank P. Deligne, F. Diamond, H. Hida, B. Mazur, R. P. Langlands, D. Ramakrishnan, T. Saito, C. Skinner, J. Tilouine, and R. Taylor for helpful comments. In particular the author is grateful to T. Saito for his encouragement during preparation of this paper. He has given the author detailed comments on earlier versions of this paper. The argument in §2 of showing the freeness is due to him.

Finally, the author is grateful to A. Wiles. Without his support, this paper would have never been written.

1. Conventions

For a field F, $G_F = \mathrm{Gal}(\bar{F}/F)$ means the absolute Galois group. For a prime ℓ, we denote the group of ℓ-power roots of unit by $\mu_{\ell^\infty}(F)$. Let $\chi_{\mathrm{cycle}} : G_F \to \mathbb{Z}_\ell^\times$ be the cyclotomic character of F, $\chi_{\mathrm{cycle}} = \chi_{\mathrm{cycle}}^{\ell} \cdot \chi_{\mathrm{cycle},\ell}$ the decomposition into prime-to ℓ and pro-ℓ-part. We denote $\chi_{\mathrm{cycle}}^{\ell}$ by ω_F.

For a non-archimedean local field F, the ring of the integers of F is denoted by o_F, with the maximal ideal m_F and the residue field k_F. A uniformizer of F is denoted by p_F. $W_F \subset G_F$ the Weil group of F, $I_F \subset G_F$ the inertia subgroup of G_F. W'_F is the Weil-Deligne group of F.

For a non-zero ideal I of o_F, subgroups $K_{11}(I) \subset K_1(I) \subset K_0(I)$ of $\mathrm{GL}_2(o_F)$ are defined by

$$\begin{aligned} K_{11}(I) &= \{g \in \mathrm{GL}_2(o_F),\ g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod I\}, \\ K_1(I) &= \{g \in \mathrm{GL}_2(o_F),\ g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \mod I\}, \\ K_0(I) &= \{g \in \mathrm{GL}_2(o_F),\ g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod I\}. \end{aligned}$$

By the local class field theory, $W_F^{\mathrm{ab}} \simeq F^\times$, where a uniformizer p_F corresponds to a geometric Frobenius element of W_F^{ab}.

The local Langlands correspondence for $\mathrm{GL}_{2,F}$ is a bijection between the isomorphism classes of F-semisimple representation ρ of the Weil-Deligne group W'_F and the isomorphism classes of admissible irreducible representation π of $\mathrm{GL}_2(F)$ [29].

When F is a global field, the set of all places (resp. finite places) is denoted by $|F|$ (resp. $|F|_f$) . For a place v of F, F_v means the local field at v. If v is a finite place, the ring integers of F_v is denoted by o_{F_v}, with the maximal ideal m_{F_v}, and a uniformizer p_v. $k(v) = o_{F_v}/m_{F_v}$ the residue field, q_v its cardinality.

For a finite set of finite places Σ, $G_\Sigma = \pi_1^{\text{ét}}(\operatorname{Spec} o_F \setminus \Sigma)$ is the Galois group of the maximal Galois extension of F which is unramified outside Σ.

$\mathbb{A}_F$ is the adéle ring of F, $\mathbb{A}_{F,f}$ is the ring of finite adeles, and $\mathbb{A}_{F,\infty}$ is the infinite part of $\mathbb{A}_F$. For a reductive group G over F, a compact open subgroup K of the adelic group $G(\mathbb{A}_{F,f})$ is factorizable if $K = \prod_v K_v$, $K_v \subset G(F_v)$. In this case, for a finite set of places Σ,

$$K_\Sigma = \prod_{v \in \Sigma} K_v,\ K^\Sigma = \prod_{v \notin \Sigma} K_v.$$

$K = K_\Sigma \cdot K^\Sigma$.

For a non-zero ideal I of o_F, we define compact open subgroups of $\mathrm{GL}_2(\mathbb{A}_{F,f})$ by

$$K(I) = \{g \in \mathrm{GL}_2(\prod_{v \in |F|_f} o_{F_v}),\ g \equiv 1 \mod I\},$$

$$\begin{aligned} K_{11}(I) &= \{g \in \mathrm{GL}_2(\prod_{v \in |F|_f} o_{F_v}),\ g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod I\}, \\ K_1(I) &= \{g \in \mathrm{GL}_2(\prod_{v \in |F|_f} o_{F_v}),\ g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \mod I\}, \\ K_0(I) &= \{g \in \mathrm{GL}_2(\prod_{v \in |F|_f} o_{F_v}),\ g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod I\}. \end{aligned}$$

For a non-archimedian local field F, let $B(F)$ be the standard Borel subgroup of $\mathrm{GL}_2(F)$ consisting of the upper triangular matrices. For two quasi-characters $\chi_i : F^\times \to \mathbb{C}^\times$ $(i = 1, 2)$ $\chi : B(F) \to \mathbb{C}^\times$ is defined by $\chi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}) = \chi_1(a)\chi_2(d)$.

The non-unitary induction $\pi(\chi_1,\chi_2)=\mathrm{Ind}_{B(F)}^{G(F)}\chi$ is given by

$$\mathrm{Ind}_{B(F)}^{G(F)}\chi=\{f:\mathrm{GL}_2(F)\to\mathbb{C},\ f(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}g)=\chi_1(a)\chi_2(d)|a|f(g)\}.$$

For a prime ℓ, we fix an isomorphism $\mathbb{C}\simeq\bar{\mathbb{Q}}_\ell$ by the axiom of choice.

Let F be a totally real field. For an infinity type (k,w) (see Definition 4.1), which satisfies $k_\iota\equiv w \mod 2$ for $\iota\in I_{F,\infty}$, $k'\in\mathbb{Z}^{I_{F,\infty}}$ is defined by the formula

$$k+2k'=(w+2)\cdot(1,\dots,1).$$

For a cupspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of discrete infinity type (k,w) such that π_f is defined over an ℓ-adic field E_λ, $\rho_{\pi,E_\lambda}:G_F\to\mathrm{GL}_2(E_\lambda)$ is the associated λ-adic representation. For a finite place $v\nmid\ell$ such that the v-component π_v is spherical, a geometric Frobenius element Fr_v satisfies

$$\mathrm{trace}\rho_{\pi,E_\lambda}(\mathrm{Fr}_v)=\alpha_v+\beta_v$$

where $(\alpha_v,\ \beta_v)$ is the Satake parameter of π_v seen as a semi-simple conjugacy class in dual group $\mathrm{GL}_2^\vee(\bar{\mathbb{Q}}_\ell)$, and π_v is a constituant of non-unitary induction $\mathrm{Ind}_{B(F_v)}^{G(F_v)}\chi_{\alpha_v,\beta_v}$. Here $\chi_{\alpha_v,\beta_v}:B(F_v)\to\bar{\mathbb{Q}}_\ell^\times$ is the unramified character defined by $\chi_{\alpha_v,\beta_v}(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix})=\alpha_v^{\mathrm{ord}_v a}\beta_v^{\mathrm{ord}_v d}$.

At infinite places, the $\mathrm{GL}_2(\mathbb{R})$-representation $D_{k,w}$ corresponds to the unitary induced representation

$$\mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}(\mu_{k,w},\nu_{k,w})_{\mathbf{u}}$$

for two characters of the split maximal torus

$$\mu_{k,w}(a)=|a|^{\frac{1}{2}-k'}(\mathrm{sgn}\,a)^{-w}$$

$$\nu_{k,w}(d)=|d|^{\frac{1}{2}-w+k'}$$

for k' satisfying $k-2+2k'=w$. This normalization, which is the $|\cdot|_v^{\frac{1}{2}}$-twist of the unitary normalization, has the merit that it preserves the field of definition. The central character of π corresponds to $\det\rho_{\pi,E_\lambda}(1)$. Our normalization is basically the same as that in [6], except one point. In [6], an arithmetic Frobenius element corresponds to a uniformizer.

The global correspondence $\pi\mapsto\rho_{\pi,E_\lambda}$ is compatible with the local Langlands correspondence for $v\nmid\ell$ ([6] théorème (A), see [50], [46] theorem 2 for the missing even degree cases), namely, if we take the F-semisimplification of $\rho_{\pi,\lambda}|_{G_{F_v}}$, this corresponds to π_v by the local Langlands correspondence normalized as above.

2. Taylor-Wiles systems

We present an abstract formulation of the argument of Taylor and Wiles [49]. Our proof is influenced by an argument of Faltings on their work ([49], Appendix). A similar method was found by Diamond independently [10].

2.1. Definition of Taylor-Wiles systems.

For a global field F, let $|F|_f$ be the set of finite places of F, and for $v\in|F|_f$, q_v means the cardinality of the residue field $k(v)$ at v.

Let o_λ be a complete noetherian local algebra with the maximal ideal m_λ. We assume that the residue field $k_\lambda=o_\lambda/m_\lambda$ is a finite field of characteristic ℓ.

Definition 2.1. *Let H be a torus over F of dimension d, X a set of finite subsets of $|F|_f$ which contains $\emptyset$. We take a pair (R, M), where R is a complete noetherian local o_λ-algebra with the residue field k_λ, and M is a finitely generated R-module.*

A Taylor-Wiles system $\{R_Q, M_Q\}_{Q\in X}$ for (R, M) consists of the following data:

TW1: *For $Q \in X$ and $v \in Q$, H is split at v, and $q_v \equiv 1 \mod \ell$. We denote by Δ_v the ℓ-Sylow subgroup of $H(k(v))$, and Δ_Q is defined as $\prod_{v\in Q} \Delta_v$ for $Q \in X$.*

TW2: *For $Q \in X$, R_Q is a complete noetherian local $o_\lambda[\Delta_Q]$-algebra with the residue field k_λ, and M_Q is an R_Q-module. For $Q = \emptyset$, $(R_\emptyset, M_\emptyset) = (R, M)$.*

TW3: *A surjection*

$$R_Q/I_Q R_Q \twoheadrightarrow R$$

of local o_λ-algebras for each $Q \in X$. Here $I_Q \subset o_\lambda[\Delta_Q]$ denotes the augmentation ideal of $o_\lambda[\Delta_Q]$. For $Q = \emptyset$, it is the identity of R.

TW4: *The homomorphism $R_Q/I_Q R_Q \to \mathrm{End}_{o_\lambda} M_Q/I_Q M_Q$ factors through R, and $M_Q/I_Q M_Q$ is isomorphic to M as an R-module.*

TW5: *M_Q is free of finite rank α as an $o_\lambda[\Delta_Q]$-module for a fixed integer $\alpha \geq 1$.*

In [49], the conditions that R_Q is Gorenstein and M_Q is a free R_Q-module are required.

Unlike Kolyvagin's Euler systems, we do not impose a functoriality when the index set grows.

Definition 2.2. *Let $\{R_Q, M_Q\}_{Q\in X}$ be a Taylor-Wiles system for (R, M) with the coefficient ring o_λ, $o_\lambda \to o'_{\lambda'}$ a local homomophism between complete noetherian local rings with finite residue fields. Then the scalar extension $\{R_{Q,o'_{\lambda'}}, M_{Q,o'_{\lambda'}}\}_{Q\in X}$ of $\{R_Q, M_Q\}_{Q\in X}$ is defined by $R_{Q,o'_{\lambda'}} = R_Q \hat{\otimes}_{o_\lambda} o'_{\lambda'}$, $M_{Q,o'_{\lambda'}} = M_Q \otimes_{o_\lambda} o'_{\lambda'}$. This is a system for $(R_{o'_{\lambda'}}, M_{o'_{\lambda'}}) = (R\hat{\otimes}_{o_\lambda} o'_{\lambda'}, M \otimes_{o_\lambda} o'_{\lambda'})$.*

2.2. **Complete intersection-freeness criterion.**

Theorem 2.3. *[Complete intersection and freeness criterion] For a Taylor-Wiles system $\{R_Q, M_Q\}_{Q\in X}$ for (R, M) and a torus H of dimension d, assume the following conditions.*

(1) *For any integer $n \in \mathbb{N}$, the subset X_n of X defined by*

$$X_n = \{Q \in X;\ v \in Q \Rightarrow q_v \equiv 1 \mod \ell^n\}.$$

is an infinite set.

(2) *$r = \sharp Q$ is independent of $Q \in X$ if $Q \neq \emptyset$.*

(3) *R_Q is generated by at most dr many elements as a complete local o_λ-algebra for all $Q \in X$.*

Then under (1)-(3),

- *R is o_λ-flat and of relative complete intersection of dimension zero.*
- *M is a free R-module.*

In particular R is isomorphic to the image T in $\mathrm{End}_{o_\lambda} M$.

Proof of Theorem 2.3. First we treat the case when o_λ is a regular local ring. We choose the following data for each element $Q \in X \setminus \{\emptyset\}$:

- An isomorphism

$$\alpha_Q : \Delta_Q \xrightarrow{\sim} \bigoplus_{v\in Q} (\mathbb{Z}/\ell^{n(v)}\mathbb{Z})^{\oplus d}$$

  as finite abelian groups. Here $\ell^{n(v)}$ is the order of the ℓ-Sylow subgroup of $k(v)^\times$.

- An isomorphism

$$\beta_Q : M_Q \xrightarrow{\sim} (o_\lambda[\Delta_Q])^\alpha$$

  as $o_\lambda[\Delta_Q]$-modules.

- A surjection $\gamma_Q : o_\lambda[[T_1,..,T_{dr}]] \twoheadrightarrow R_Q$ as complete local o_λ-algebras.

The existence of β_Q and γ_Q is assured by TW5 and 2.3, (3).

By the assumption (1), X_n is an infinite set.

For $\Gamma = \mathbb{Z}_\ell^{\oplus dr}$, let S_∞ be the complete group ring $o_\lambda[[\Gamma]]$, and I_∞ the augmentation ideal.

For an element $Q \in X_n$, where X_n is defined in 2.3 (1), α_Q induces an isomorphism $\Delta_Q/\ell^n\Delta_Q \xrightarrow{\sim} \Gamma/\ell^n\Gamma$ of abelian groups, so the o_λ-algebra isomorphism

$$\epsilon_{Q,n} : o_\lambda[\Delta_Q]/J_{Q,n} \xrightarrow{\sim} S_n$$

is induced. Here $J_{Q,n}$ is the ideal of $o_\lambda[\Delta_Q]$ generated by m_λ^n and $\delta^{\ell^n} - 1$ for $\delta \in \Delta_Q$, $S_n = (o_\lambda/m_\lambda^n)[\Gamma/\ell^n\Gamma]$, which is viewed as a quotient of S_∞. $I_n = I_\infty S_n$ is the augmentation ideal of S_n as the group ring of $\Gamma/\ell^n\Gamma$ over o_λ/m_λ^n.

β_Q and $\epsilon_{Q,n}$ induce an isomorphism

$$M_Q/J_{Q,n}M_Q \xrightarrow{\sim} S_n^\alpha$$

as S_n-modules. $R_{Q,n}$ is defined as the image of $R_Q/J_{Q,n}R_Q$ in $\mathrm{End}_{S_n}(M_Q/J_{Q,n}M_Q) \xrightarrow{\sim} \mathrm{End}_{S_n} S_n^\alpha$, which is viewed as an S_n-algebra by $\epsilon_{Q,n}$.

Now we apply the method of Taylor and Wiles to construct a projective system which approximates a power series ring. For an integer $n \in \mathbb{N}$ and $Q \in X_n$, we consider the couple $((R_{Q,n}, \iota_{Q,n}), p_{Q,n})$ defined as follows:

(∗) $R_{Q,n}$ is a finite S_n-algebra with a faithful action on S_n^α by $\iota_{Q,n} : R_{Q,n} \hookrightarrow \mathrm{End}_{S_n} S_n^\alpha$.

(∗∗) $p_{Q,n} : o_\lambda[[T_1 \ldots, T_{dr}]] \twoheadrightarrow R_{Q,n}$ is the surjective homomorphism $o_\lambda[[T_1 \ldots, T_{dr}]] \xrightarrow{\gamma_Q} R_Q \to R_{Q,n}$ as local o_λ-algebras.

The isomorphism classes of the couples $((R_{Q,n}, \iota_{Q,n}), p_{Q,n})$ are finite as Q varies in X_n, since the cardinality of $R_{Q,n}$ is bounded. Thus there is a sequence $\{Q(n)\}_{n\in\mathbb{N}}$ which satisfies the following properties:

- $Q(n)$ is an element of X_n.
- For $m \le n$, $((R_{Q(n),m}, \iota_{Q(n),m}), p_{Q(n),m})$ is isomorphic to $((R_{Q(m),m}, \iota_{Q(m),m}), p_{Q(m),m})$.

For the sequence $\{Q(n)\}_{n\in\mathbb{N}}$ thus obtained, $((R_{Q(n),n}, \iota_{Q(n),n}), p_{Q(n),n})_{n\in\mathbb{N}}$ forms a projective system. The transition map

$$R_{Q(n+1),n+1} \longrightarrow R_{Q(n+1),n} \simeq R_{Q(n),n}$$

is surjective for any $n \in \mathbb{N}$, since $R_{Q(n+1),n}$ is a quotient of $R_{Q(n+1),n+1} \otimes_{S_{n+1}} S_n$.

By taking the projective limit, we define

$$P = \varprojlim_n R_{Q(n),n},$$

which has a structure of S_∞-algebra. By (∗∗), there is a surjection $o_\lambda[[T_1,..,T_{dr}]] \twoheadrightarrow P$. By the definition, P has a faithful non-zero module

$$L = \varprojlim_n S_n{}^{\oplus\alpha} \simeq \varprojlim_n M_{Q(n)}/J_{Q(n),n}M_{Q(n)},$$

which is S_∞-free and finitely generated over S_∞.

Lemma 2.4. *For the o_λ-algebra P and the P-module L thus defined, the following holds:*

(1) *The local o_λ-homomorphism $o_\lambda[[T_1,..,T_{dr}]] \twoheadrightarrow P$ is an isomorphism.*
(2) *P is S_∞-flat.*
(3) *L is a non-zero free P-module.*

Proof of Lemma 2.4. We prove (1). The o_λ-algebra homomorphism $S_\infty \to P$ is injective since any element in the kernel must annihilate a non-zero free S_∞-module L. Since P is an S_∞-subalgebra of $\mathrm{End}_{S_\infty} L$, P is a finite S_∞-algebra. It follows that $\dim P = dr + \dim o_\lambda$, and the surjection $o_\lambda[[T_1, .., T_{dr}]] \twoheadrightarrow P$ must be an isomorphism.

Since P is regular and finite over S_∞, (2) follows from [18], Chap. IV, proposition 6.1.5.

L is finitely generated and flat as an S_∞-module. Then L is a finitely generated P-module, and P-free by the following sublemma.

Sublemma 2.5. *Let A, B be regular local rings, and M a finitely generated B-module. If B is finite flat over A, and M is A-flat, then M is B-flat.*

Proof of Sublemma 2.5. By the assumption, $\mathrm{depth}_B\, M = \mathrm{depth}\, A = \mathrm{depth}\, B$. Since the projective dimension of M is finite, by the Auslander-Buchsbaum formula [17], Chap.0, proposition 17.3.4, $\mathrm{proj.\,dim}_B\, M = \mathrm{depth}\, B - \mathrm{depth}_B\, M = 0$ and hence M is projective. □

□

By Lemma 2.4 (2),

$$\tilde{R} = P/I_\infty P$$

is finite flat as an $S_\infty/I_\infty = o_\lambda$-module.

The ideal $I = I_\infty P$ of P is generated by dr many elements. This means that $\mathrm{ht}\, I \leq dr$. Since P is catenary, $\dim \tilde{R} = \dim P - \mathrm{ht}\, I \geq dr + \dim o_\lambda - dr = \dim o_\lambda$. On the other hand, $\dim \tilde{R} \leq \dim o_\lambda$ because $\tilde{R}$ is finitely generated as an o_λ-module. Thus we have the equalities $\dim \tilde{R} = \dim o_\lambda$ and $\mathrm{ht}\, I = dr$. This implies that dr is the minimal number of generators of I, so $\tilde{R}$ is of relative complete intersection over o_λ.

By Lemma 2.4 (3), $\tilde{R}$-module $L/I_\infty L$ is a non-zero free module. $L_\infty/I_\infty L$ is naturally isomorphic to

$$\varprojlim_n M_{Q(n)}/(m_\lambda^n, I_{Q(n)}\ )M_{Q(n)} \simeq \varprojlim_n M/m_\lambda^n = M$$

by the definition and TW4. Since the pair $(\tilde{R}, M)$ has the desired complete intersection-freeness property, to conclude the proof of Theorem 2.3 when o_λ is regular, it is sufficient to identify R and $\tilde{R}$.

Lemma 2.6. (1) *The canonical homomorphism $P \otimes_{S_\infty} S_n \twoheadrightarrow R_{Q(n),n}$ is an isomorphism, and $M_{Q(n),n}/J_{Q(n),n}M_{Q(n),n}$ is a non-zero free $P \otimes_{S_\infty} S_n$-module.*

(2) *$\tilde{R}$ is a quotient of R.*

Proof of Lemma 2.6. $L \otimes_{S_\infty} S_n = M_{Q(n)}/J_{Q(n),n}M_{Q(n)}$ is a free $P \otimes_{S_\infty} S_n$-module by Lemma 2.4, (3). Since the $P \otimes_{S_\infty} S_n$-action on $L \otimes_{S_\infty} S_n$ factors through the quotient $R_{Q(n),n}$, (1) is shown.

Since $M_{Q(n)}/J_{Q(n),n}M_{Q(n)}$ is $R_{Q(n),n}$-free by (1),

$$M_{Q(n)}/J_{Q(n),n}M_{Q(n)} \otimes_{S_n} S_n/I_n = M/m_\lambda^n M,$$

is a free $R_{Q(n),n} \otimes_{S_n} S_n/I_n = (P \otimes_{S_\infty} S_n) \otimes_{S_n} S_n/I_n = \tilde{R}/m_\lambda^n \tilde{R}$-module. Thus $\tilde{R}/m_\lambda^n \tilde{R}$ coincides with the image of $R_{Q(n),n}$ in $\mathrm{End}_{o_\lambda/m_\lambda^n} M/m_\lambda^n M$. The $R_{Q(n)}$-action on $M_{Q(n)}/I_{Q(n)}M_{Q(n)} = M$ factors through R by TW4, and hence $\tilde{R}/m_\lambda^n \tilde{R}$ is regarded as a quotient of R. By passing to the projective limit, (2) is shown. □

Fix an integer $N \geq 1$. Since the transition maps in the projective system $\{R_{Q(n),n}\}_{n \in \mathbb{N}}$ are surjective, there is some integer $n \geq N$ such that

$$R_{Q(n),n}/m_{R_{Q(n),n}}^N \simeq P/m_P^N \simeq o_\lambda[[T_1, .., T_{dr}]]/(m_\lambda, T_1, .., T_{dr})^N.$$

On the other hand, $R_{Q(n)}/m^N_{R_{Q(n)}}$ is a quotient of $o_\lambda[[T_1,..,T_{dr}]]/(m_\lambda,T_1,..,T_{dr})^N$ by the condition 2.3, (3). As $R_{Q(n),n}$ is a quotient of $R_{Q(n)}$, $R_{Q(n)}/m^N_{R_{Q(n)}} = R_{Q(n),n}/m^N_{R_{Q(n),n}}$ holds.

We define the o_λ-algebra $\tilde{R}_{Q(n)}$ by $\tilde{R}_{Q(n)} = R_{Q(n)}/I_{Q(n)}R_{Q(n)}$. Since $R_{Q(n),n}/I_{Q(n)}R_{Q(n),n}$ is isomorphic to $\tilde{R}/m^n_\lambda\tilde{R}$ by Lemma 2.6 (1), it follows that $\tilde{R}_{Q(n)}/m^N_{\tilde{R}_{Q(n)}} \xrightarrow{\sim} \tilde{R}/m^N_{\tilde{R}}$. The isomorphism $\tilde{R}_{Q(n)}/m^N_{\tilde{R}_{Q(n)}} \xrightarrow{\sim} \tilde{R}/m^N_{\tilde{R}}$ is factorized as

$$\tilde{R}_{Q(n)}/m^N_{\tilde{R}_{Q(n)}} \twoheadrightarrow R/m^N_R \twoheadrightarrow \tilde{R}/m^N_{\tilde{R}}.$$

This implies that $R/m^N_R \xrightarrow{\sim} \tilde{R}/m^N_{\tilde{R}}$, and hence $R \xrightarrow{\sim} \tilde{R}$ by passing to the limit with respect to N.

We prove the general case. Let $k = k_\lambda$ be the residue field of o_λ, and $(R_Q \otimes_{o_\lambda} k, M_Q \otimes_{o_\lambda} k)_{Q\in X}$ the Taylor-Wiles system for $(R \otimes_{o_\lambda} k, M \otimes_{o_\lambda} k)$ obtained by the scalar extension. Since k is regular, we have shown that $R \otimes_{o_\lambda} k$ is a finite k-algebra of complete intersection, and $M \otimes_{o_\lambda} k$ is $R \otimes_{o_\lambda} k$-free. In particular this implies that R is finite over o_λ.

We take an integer $\beta \geq 1$ and a surjective R-homomorphism $g : R^{\oplus\beta} \twoheadrightarrow M$ which induces an isomorphism $(R/m_\lambda R)^{\oplus\beta} \simeq M/m_\lambda M$. Let M' be the kernel of g, which is finitely generated as an o_λ-module. Since M is o_λ-flat, $\mathrm{Tor}_1^{o_\lambda}(M, k_\lambda) = \{0\}$, and the sequence

$$0 \longrightarrow M' \otimes_{o_\lambda} k \longrightarrow (R^{\oplus\beta}) \otimes_{o_\lambda} k \xrightarrow{g\otimes \mathrm{id}_k} M \otimes_{o_\lambda} k \longrightarrow 0$$

is exact. By Nakayama's lemma, $M' = \{0\}$, and g is an isomorphism, that is, M is R-free.

By the o_λ-flatness of M, R is o_λ-flat. Since R is o_λ-flat and $R \otimes_{o_\lambda} k$ is of complete intersection, R is of relative complete intersection over o_λ. $\square$

2.3. **Hierarchy of complete intersection and freeness.** In this subsection, we give a formulation of a level raising argument of [51].

Let E_λ be an ℓ-adic field, $o_\lambda = o_{E_\lambda}$ the integer ring, m_λ the maximal ideal.

Definition 2.7. (1) *An admissible quintet is a quintet $(R, T, \pi, M, \langle\ ,\ \rangle)$, where R is a complete noetherian local o_λ-algebra, T is a finite flat o_λ-algebra, $\pi : R \to T$ is a surjective o_λ-algebra homomorphism, M is a faithful finitely generated T-module which is o_λ-free, and $\langle\ ,\ \rangle : M \otimes_{o_\lambda} M \to o_\lambda$ is a perfect pairing which induces $M \simeq \mathrm{Hom}_{o_\lambda}(M, o_\lambda)$ as a T-module.*

(2) *An admissible quintet $(R, T, \pi, M, \langle\ ,\ \rangle)$ is distinguished if R is of complete intersection, and M is a non-zero free R-module (it follows that π is an isomorphism).*

(3) *An admissible morphism from $(R', T', \pi', M', \langle\ ,\ \rangle')$ to $(R, T, \pi, M, \langle\ ,\ \rangle)$ is a triple (α, β, ξ). Here $\alpha : R' \to R$, $\beta : T' \to T$ are surjective o_λ-algebra homomorphisms making the following diagram*

$$\begin{array}{ccc} R' & \xrightarrow{\alpha} & R \\ {\scriptstyle\pi'}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ T' & \xrightarrow{\beta} & T \end{array}$$

commutative, and $\xi : M \hookrightarrow M'$ is an injective T'-homomorphism onto an o_λ-direct summand.

A Taylor-Wiles system gives rise to a distinguished admissible quintet for a suitably chosen pairing on M under the condition of Theorem 2.3. Note that we do not assume that the restriction of $\langle\ ,\ \rangle'$ to $\xi(M)$ is $\langle\ ,\ \rangle$ in the definition of admissible morphisms.

For an admissible morphism (α, β, ξ) from $(R', T', \pi', M', \langle\ ,\ \rangle')$ to $(R, T, \pi, M, \langle\ ,\ \rangle)$, there is a criterion for $(R', T', \pi', M', \langle\ ,\ \rangle')$ to be a distinguished quintet under the condition that $(R, T, \pi, M, \langle\ ,\ \rangle)$ is distinguished.

For a finitely generated o_λ-module L, we denote $\mathrm{Hom}_{o_\lambda}(L, o_\lambda)$ by $L^\vee$.

By the perfect pairings on M and M', we make the identifications $M \simeq M^\vee$ and $M' \simeq (M')^\vee$, and $\xi^\vee$ is defined as

$$\xi^\vee : M' \xrightarrow{\sim} (M')^\vee \longrightarrow M^\vee \xrightarrow{\sim} M,$$

such that

$$\langle\ \xi(x),\ y\ \rangle' = \langle\ x,\ \xi^\vee(y)\ \rangle \quad (\forall x \in M,\ \forall y \in M')$$

holds.

For any noetherian local o_λ-algebra R with an o_λ-algebra homomorphism $f : R \to o_\lambda$, one attaches a numerical invariant to it, following [31]:

Consider the annihilator $\mathrm{Ann}_R(\ker\ f)$ of $\ker\ f$ in R. Define an ideal η_f of o_λ by

$$\eta_f = \text{ the image of } \mathrm{Ann}_R(\ker\ f) \text{ by } f.$$

When R is a finite flat Gorenstein o_λ-algebra, $\mathrm{Ann}_R(\ker f)$ is generated by the image of 1 of the o_λ-module homomorphism

$$o_\lambda \xrightarrow{f^\vee} \omega_R \xrightarrow{\sim} R,$$

where $\omega_R = R^\vee$ is the dualizing module of R, and η_f is generated by the image of 1 of

$$o_\lambda \xrightarrow{f^\vee} \omega_R \xrightarrow{\sim} R \longrightarrow o_\lambda.$$

We fix an o_λ-algebra homomorphism $f_T : T \to o_\lambda$. f_R (resp.$f_{T'}$, resp. $f'_{R'}$) is defined as $f_T \circ \pi$ (resp. $f_T \circ \beta$, resp. $f_T \circ \beta \circ \pi'$).

Theorem 2.8. *(abstract level raising formalism) For an admissible morphism between admissible quintets* $(R', T', \pi', M', \langle\ ,\ \rangle') \to (R, T, \pi, M, \langle\ ,\ \rangle)$, *we assume the following conditions:*

1. $(R, T, \pi, M, \langle\ ,\ \rangle)$ *is distinguished.*
2. T *and* T' *are reduced,* $M' \otimes_{o_\lambda} E_\lambda$ *is* $T' \otimes_{o_\lambda} E_\lambda$*-free, and the rank is equal to* $\mathrm{rank}_T M$.
3. *The equality*
$$\xi^\vee \circ \xi(M) = \Delta \cdot M$$
holds for some non-zero divisor Δ *in* T.
4. *The inequality*
$$\mathrm{length}_{o_\lambda} \ker f_{R'}/(\ker f_{R'})^2 \leq \mathrm{length}_{o_\lambda} \ker f_R/(\ker f_R)^2 + \mathrm{length}_{o_\lambda} o_\lambda/\ f_T(\Delta) o_\lambda$$
holds.

Then $(R', T', \pi', M', \langle\ ,\ \rangle')$ *is also distinguished, that is,* $\pi' : R' \simeq T'$, R' *is of complete intersection, and* M' *is* T'*-free.*

Remark 2.9. *The freeness of* M' *follows from theorem 2.4 of* [10]*. The argument for the freeness in the following is due to T. Saito.*

We will show a slightly stronger statement than Theorem 2.8. The deduction of Theorem 2.8 from Theorem 2.10 is left to the reader.

Theorem 2.10. *(Lifting theorem) For an admissible morphism between admissible quintets* $(R', T', \pi', M', \langle\ ,\ \rangle') \to (R, T, \pi, M, \langle\ ,\ \rangle)$, *we assume the following conditions:*

1. $(R, T, \pi, M, \langle\ ,\ \rangle)$ *is distinguished.*
2. $(M'/\xi(M))_{E_\lambda}$ *does not have any non-zero subquotient which is a* T_{E_λ}*-module.*

(3) *The equality*

$$\xi^\vee \circ \xi(M) = \Delta \cdot M$$

holds for some element Δ *in* T.

(4) $\eta_{f_T} \neq \{0\}$, *and the inequality*

$$\mathrm{length}_{o_\lambda} \ker f_{R'}/(\ker f_{R'})^2 \leq \mathrm{length}_{o_\lambda} \ker f_R/(\ker f_R)^2 + \mathrm{length}_{o_\lambda} o_\lambda/\ f_T(\Delta)o_\lambda$$

holds.

Then $\pi' : R' \to T'$ *is an isomorphism,* R *is of complete intersection, and there is a* T'-*free direct summand* F' *of* M' *such that the restriction of* $\langle\ ,\ \rangle'$ *to* F' *is perfect, and* $\mathrm{rank}_{T'} F' = \mathrm{rank}_T M$ *holds.*

Lemma 2.11. *Assume that we are given an admissible morphism* $(R', T', \pi', M', \langle\ ,\ \rangle') \to (R, T, \pi, M, \langle\ ,\ \rangle)$ *which satisfies conditions (1) and (2) of 2.10. If* $\xi^\vee \cdot \xi(M) = \Delta \cdot M$ *for some element* Δ *in* T, *then*

$$\mathrm{length}_{o_\lambda} o_\lambda/\eta_{f_{T'}} \geq \mathrm{length}_{o_\lambda} o_\lambda/f_T(\Delta)o_\lambda + \mathrm{length}_{o_\lambda} o_\lambda/\eta_{f_T}$$

holds.

Proof of Lemma 2.11. First we show that $\mathrm{Ann}_{T'}(\ker f_{T'})M' \subset \xi(\mathrm{Ann}_M(\ker f_T))$. Any element f in $\ker f_{T'}$ acts as zero on $\mathrm{Ann}_{T'}(\ker f_{T'})M'$, so that $\mathrm{Ann}_{T'}(\ker f_{T'})M'$ admits a structure of $T'/\ker f_{T'} \simeq T/\ker f_T$-module. Since $(M'/\xi(M))_{E_\lambda}$ does not contain any non-zero T_{E_λ}-modules by 2.10 (2), $\mathrm{Ann}_{T'}(\ker f_{T'})M' \subset \xi(M)$ because $\xi(M)$ is an o_λ-direct summand of M'. If we identify $\mathrm{Ann}_{T'}(\ker f_{T'})M'$ as a submodule N of M, any element in N is annihilated by $\ker f_T$, and N is contained in $\mathrm{Ann}_M(\ker f_T)$.

By 2.10 (1), M is T-free, so we have the equality $\mathrm{Ann}_M(\ker f_T) = (\mathrm{Ann}_T \ker f_T)M$. By applying $\xi^\vee$, we obtain $\xi^\vee(\mathrm{Ann}_{T'}(\ker f_{T'})M') \subset \xi^\vee \circ \xi((\mathrm{Ann}_T \ker f_T)M) = (\Delta \cdot \mathrm{Ann}_T \ker f_T)M$.

$h : M' \otimes_{f_{T'}} o_\lambda \to M \otimes_{f_{T'}} o_\lambda = M \otimes_{f_T} o_\lambda$ is surjective. Since $M \otimes_{f_T} o_\lambda$ is non-zero and o_λ-free, we find a surjective o_λ-homomorphism $g : M \otimes_{f_T} o_\lambda \twoheadrightarrow L$ such that $L \simeq o_\lambda$. Because o_λ is PID, there is an element x of M' such that $g \circ h(x \otimes_{f_{T'}} 1)$ generates L. The image of $\mathrm{Ann}_{T'}(\ker f_{T'})x$ in L is $\eta_{f_{T'}} L$.

$g \circ h(\mathrm{Ann}_{T'}(\ker f_{T'})x \otimes_{f_{T'}} 1) \subset g((\Delta \cdot \mathrm{Ann}_T \ker f_T)M \otimes_{f_T} o_\lambda) = f_T(\Delta) \cdot \eta_{f_T} g(M \otimes_{f_T} o_\lambda) = f_T(\Delta) \cdot \eta_{f_T} L$. The claim is shown.

□

Proof of Theorem 2.10 . First we show that R' is of complete intersection, and $\pi' : R' \to T'$ is an isomorphism.

Since T is of complete intersection, the equality

$$\mathrm{length}_{o_\lambda} o_\lambda/\eta_{f_T} = \mathrm{length}_{o_\lambda} \ker f_T/(\ker f_T)^2$$

holds by [31].

We show $\xi^\vee_{E_\lambda} \circ \xi_{E_\lambda}$ is an isomorphism. By assumption 2.10, (2), the restriction of $\xi^\vee : (M')^\vee \to M^\vee$ to $(M'/\xi(M))^\vee$ is zero. So the induced map $\xi^\vee_{E_\lambda}|_{\xi(M)_{E_\lambda}}$ must be surjective. By compairing the dimension, it is an isomorphism.

In particular Δ is invertible in T_{E_λ}, and $f_T(\Delta) \neq 0$.

By assumption 2.10, (4) and Lemma 2.11, we have

$$(*_1) \qquad \begin{aligned} \mathrm{length}_{o_\lambda} \ker f_{R'}/(\ker f_{R'})^2 &\leq \mathrm{length}_{o_\lambda} \ker f_R/(\ker f_R)^2 + \mathrm{length}_{o_\lambda} o_\lambda/\ f_T(\Delta)o_\lambda \\ &\leq \mathrm{length}_{o_\lambda} o_\lambda/\eta_{f_{T'}}. \end{aligned}$$

Then the claim follows from the isomorphism criterion of Lenstra [31].

As a consequence, we have the equality

$$(*_2) \qquad \mathrm{length}_{o_\lambda} o_\lambda/\eta_{f_{T'}} = \mathrm{length}_{o_\lambda} o_\lambda/\eta_{f_T} + \mathrm{length}_{o_\lambda} o_\lambda/\ f_T(\Delta)o_\lambda,$$

since the equality holds in $(*_1)$.

We construct a T'-free direct summand of M'. There is a canonical T'-homomorphism

$$\beta^\vee : \omega_T \longrightarrow \omega_{T'}$$

by the duality.

Since T and T' are complete intersections, we choose isomorphisms $\delta_T : T \overset{\sim}{\to} \omega_T$ and $\delta_{T'} : T' \overset{\sim}{\to} \omega_{T'}$ as T'-modules. We define Δ_0 by

$$(*_3) \qquad \Delta_0 : T \xrightarrow[\sim]{\delta_T} \omega_T \xrightarrow{\beta^\vee} \omega_{T'} \xrightarrow[\sim]{\delta_{T'}^{-1}} T' \xrightarrow{\beta} T.$$

Δ_0 is a multiplication by an element δ_0 in T since it is a T'-homomorphism.

By assumption 2.10, (1), M is T-free. Let $(e_j)_{j\in J}$ be a basis of M, which yields an isomorphism $e : F = T^{\oplus c} \overset{\sim}{\to} M$. For $j \in J$, we choose an element $\tilde{e}_j$ of M' such that $\beta(\tilde{e}_j) = e_j$. $(\tilde{e}_j)_{j\in J}$ gives a T'-homomorphism $e' : F' \to M'$, where $F' = T'^{\oplus c}$. The diagram

$$\begin{tikzcd} F' \arrow[r, "\beta^{\oplus c}"] \arrow[d, "e'"'] & F \arrow[d, "e"] \\ M' \arrow[r, "\xi"] & M \end{tikzcd}$$

is commutative.

Let $B : F \to F'$ be a T'-homomorphism defined by

$$B : F \xrightarrow[\sim]{(\delta_T)^{\oplus c}} (\omega_T)^{\oplus c} \xrightarrow{(\beta^\vee)^{\oplus c}} (\omega_{T'})^{\oplus c} \xrightarrow[\sim]{(\delta_{T'}^{-1})^{\oplus c}} F'.$$

By the definition, the composition $F \overset{B}{\to} F' \overset{\beta^{\oplus c}}{\to} F$ is $(\Delta_0)^{\oplus c}$, and hence it is the multiplication by δ_0.

Consider the T'-submodule $(e' \circ B)(F)$ of M'. By assumption 2.10, (2), $\mathrm{coker}(\xi_{E_\lambda})$ does not have any non-zero T'_{E_λ}-subquotients which are T_{E_λ}-modules, which implies that $(e' \circ B)(T)$ is a T'-submodule of $\xi_{E_\lambda}(M_{E_\lambda})$. Since $\xi(M)$ is an o_λ-direct summand of M', $(e' \circ B)(T) \subset \xi(M)$, and there is a T'-homomorphism $f : F \to M$ that makes the following diagram commutative:

$$(*_4) \qquad \begin{tikzcd} F \arrow[r, "B"] \arrow[d, "f"'] & F' \arrow[r, "\beta^{\oplus c}"] \arrow[d, "e'"'] & F \arrow[d, "e"'] \\ M \arrow[r, "\xi"] & M' \arrow[r, "\xi^\vee"] & M. \end{tikzcd}$$

Since Δ is a non-zero divisor in T, there is a T-automorphism μ of M such that $\xi^\vee \circ \xi = \Delta \cdot \mu$.

We have

$$(*_5) \qquad e \circ (\beta^{\oplus c}) \circ B = e \circ ((\Delta_0)^{\oplus c}) = \delta_0 \cdot e = \xi^\vee \circ \xi \circ f = \Delta \cdot \mu \circ f.$$

This implies that $(\delta_0) \subset (\Delta)$ as ideals of T by evaluating $(*_5)$ at some e_i, and

$$\delta_0 = \alpha \cdot \Delta$$

for some $\alpha \in T$. We show that α is a unit.

Since T and T' are Gorenstein, η_{f_T} and $\eta_{f_{T'}}$ are generated by the images of 1 by $o_\lambda \overset{(f_T)^\vee}{\to} \omega_T \underset{\sim}{\overset{\delta_T^{-1}}{\to}} T \overset{f_T}{\to} o_\lambda$ and $o_\lambda \overset{(f_{T'})^\vee}{\to} \omega'_T \underset{\sim}{\overset{\delta_{T'}^{-1}}{\to}} T' \overset{f_{T'}}{\to} o_\lambda$, respectively. By the definition of δ_0 $(*_3)$, we have $\eta_{f_{T'}} = f_T(\delta_0) \cdot \eta_{f_T}$, and hence

$$(*_6) \qquad \mathrm{length}_{o_\lambda} o_\lambda/\eta_{f_{T'}} = \mathrm{length}_{o_\lambda} o_\lambda/\eta_{f_T} + \mathrm{length}_{o_\lambda} o_\lambda/\ f_T(\delta_0)o_\lambda$$

holds. Comparing $(*_6)$ with $(*_2)$, $f_T(\delta_0)o_\lambda = f_T(\Delta)o_\lambda$. This implies that $f_T(\alpha)$ is a unit in o_λ, then α must be a unit in T. Replacing Δ by $\alpha \cdot \Delta$, we may assume that $\delta_0 = \Delta$.

From $(*_5)$, $f : F \to M$ is an isomorphism since $\delta_0 = \Delta$. Taking the dual of $(*_4)$, we have a commutative diagram

$$\begin{array}{ccccc}
F & \xrightarrow{B} & F' & \xrightarrow{\beta^{\oplus c}} & F \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle e'}\big\downarrow & & {\scriptstyle e}\big\downarrow \\
M & \xrightarrow{\xi} & M' \simeq M'^{\vee} & \xrightarrow{\xi^{\vee}} & M \simeq M^{\vee} \\
 & & {\scriptstyle (e')^{\vee}}\big\downarrow & & {\scriptstyle f^{\vee}}\big\downarrow \\
 & & (F')^{\vee} & \xrightarrow{B^{\vee}} & F^{\vee}.
\end{array}$$

By the definition of B, $B^{\vee}$ induces a T-isomorphism $F^{\vee} \simeq (F')^{\vee} \otimes_{T'} T$, and is identified with $(F')^{\vee} \to (F')^{\vee} \otimes_{T'} T$.

Since $f^{\vee}$ and e are isomorphisms and $\beta^{\oplus c}$ is surjective, $((e')^{\vee} \circ e')(F') + I(F')^{\vee} = (F')^{\vee}$, where $I = \ker(T' \to T)$. By Nakayama's lemma, $((e')^{\vee} \circ e')(F') = (F')^{\vee}$, and hence $(e')^{\vee} \circ e'$ is surjective. Since the surjectivity implies the injectivity for a homomorphism between finite free modules of the same rank over a commutative ring, $(e')^{\vee} \circ e'$ is an isomorphism, and $e'(F')$ is a T'-direct summand of M'. The identification $(M')^{\vee} = \mathrm{Hom}_{o_\lambda}(M', o_\lambda) \simeq M'$ is made by the pairing $\langle\ ,\ \rangle'$, and the composition of $F' \xrightarrow{e'} M' \xrightarrow{\sim} (M')^{\vee} \xrightarrow{(e')^{\vee}} (F')^{\vee}$ is an isomorphism. Thus the restriction of the pairing to F' is perfect.

□

3. Galois deformations

For a prime ℓ and an ℓ-adic field E_λ with the integer ring o_{E_λ}, the maximal ideal of o_{E_λ} is denoted as λ, and $k_\lambda = o_{E_\lambda}/\lambda$ is the residue field. For a perfect field k of characteristic ℓ, $W(k)$ denotes the Witt ring of k.

In this section, we discuss deformation properties of a Galois representation

$$\bar{\rho} : G_F \longrightarrow \mathrm{GL}_2(k),$$

where k is a finite field of characteristic ℓ, in particular when F is a local field.

For a local field F, we denote the integer ring by o_F and the residue field by k_F. Let p be the residue characteristic, p_F a uniformizer, and $q = \sharp k_F$. By G_F and I_F, we denote the absolute Galois group of F and the inertia subgroup, respectively.

3.1. **Representations outside ℓ.** In this subsection, assume that $\ell \neq p$. The most basic classification is whether $\bar{\rho}$ is absolutely irreducible or absolutely reducible.

In the absolutely irreducible case, there are two subcases:

- 0_E. $\bar{\rho}|_{I_F}$ is absolutely irreducible, $q \equiv -1 \mod \ell$. In this case, $\bar{\rho}$ is isomorphic to $\mathrm{Ind}_{G_{\tilde{F}}}^{G_F} \bar{\psi}$, where $\tilde{F}$ is the degree two unramified extension of F, $\bar{\psi} : G_{\tilde{F}} \to k^{\times}$ is a character which does not extend to G_F. We call this case *exceptional.*
- 0_{NE}. $\bar{\rho}$ is absolutely irreducible but not of type 0_E.

In the absolutely reducible case, by extending k if necessary, we may assume that $\bar{\rho}$ is reducible over k. There is a character $\bar{\mu} : G_F \to k^\times$ such that the twist $\bar{\rho} \otimes \bar{\mu}^{-1}$ has the non-trivial I_F-fixed part. There are three subcases:

1_{SP}. For some character $\bar{\mu} : G_F \to k^\times$, $(\bar{\rho} \otimes \bar{\mu}^{-1})^{I_F}$ is one dimensional, and the semi-simplification $(\bar{\rho} \otimes \bar{\mu}^{-1})^{\mathrm{ß}}$ is unramified.
1_{PR}. For some characters $\bar{\mu}, \bar{\mu}'$, $\bar{\rho} \simeq \bar{\mu} \oplus \bar{\mu}'$, and $\bar{\mu}'/\bar{\mu}$ is ramified.
2_{PR}. For some character $\bar{\mu}$, $\bar{\rho} \otimes \bar{\mu}^{-1}$ is unramified.

Note that this classification is stable under a twist by a character $\bar{\rho} \mapsto \bar{\rho} \otimes \bar{\chi}$.

Definition 3.1. *Assume that $\ell \neq p$. Let $\bar{\rho} : G_F \to \mathrm{GL}_2(k)$ be a Galois representation where k is a finite field.*

(1) *For a reducible G_F-representation $\rho : G_F \to \mathrm{GL}_2(k)$, a character $\bar{\mu} : G_F \to k^\times$ is called a twist character for $\bar{\rho}$ if $(\bar{\rho} \otimes \bar{\mu}^{-1})^{I_F} \neq \{0\}$. The restriction $\bar{\kappa} = \bar{\mu}|_{I_F}$ to I_F is called a twist type of $\bar{\rho}$.*
(2) *In the case of 0_E, assume that $\bar{\rho}|_{G_{\tilde{F}}}$ is reducible over k. A character $\bar{\psi}$ which appear as a constituant of $\bar{\rho}|_{G_{\tilde{F}}}$ is called an inertia character of $\bar{\rho}$. The restriction $\bar{\psi}|_{I_{\tilde{F}}}$ is called an inertia type of $\bar{\rho}$.*

Since $\bar{\kappa}$ extends to a character of G_F, it is identified with a character of the G_F-coinvariant $(I_F^{\mathrm{ab}})_{G_F}$, which is canonically isomorphic to $o_F^\times$ by the local class field theory.

In the cases of 1_{SP}, and 2_{PR}, a twist type $\bar{\kappa}$ is unique for a given $\bar{\rho}$. In the case of 1_{PR}, there are two choices of twist types. We choose one twist type in this case.

Remark 3.2. (1) *The notation " case A_a" has the following meaning. The number A is the maximum of dimensions of the I_F-fixed part of twists of $\bar{\rho}$. E (resp. NE, PR, SP) means exceptional (resp. non-exceptional, principal series, special). In the cases of PR and SP, typical situations arise from admissible irreducible representations of these types by the local Lanlglands correspondence.*
(2) *For $F = \mathbb{Q}_p$ ($p \neq \ell$), 0_{NE}, 1_{SP}, and 1_{PR} cases belong to case C, A, B of* [51], *respectively. 0_E-case is treated by Diamond* [9].

3.2. **Deformations outside ℓ.** Let o_λ be a complete noetherian local ring with the maximal ideal m_{o_λ} and the finite residue field k_{o_λ} of characteristic ℓ.

By $\mathscr{C}_{o_\lambda}^{\mathrm{noeth}}$ (resp. $\mathscr{C}_{o_\lambda}^{\mathrm{artin}}$), we mean the category of the complete noetherian (resp. artinian) local o_λ-algebras A with the maximal ideal m_A such that the residue field k_A is k_{o_λ}, where morphisms are local o_λ-algebra homomorphisms. By a deformation of $\bar{\rho} : G_F \to \mathrm{GL}_n(k_{o_\lambda})$, we mean a continuous representation

$$\rho : G_F \longrightarrow \mathrm{GL}_n(A)$$

such that $\rho \mod m_A = \bar{\rho}$. Two deformations ρ and ρ' of $\bar{\rho}$ is isomorphic if there is a G_F-isomorphism $f : \rho \simeq \rho'$ such that $f \mod m_A$ is the identity. In this paper, the case of $n \leq 2$ is considered.

Definition 3.3. *(finite and unrestricted deformations)*

(1) *A deformation ρ of $\bar{\rho}$ without any conditions is called an unrestricted deformation.*
(2) *Assume that $\ell \neq p$. Let $\chi : G_F \to A^\times$ be a deformation of $\bar{\chi} : G_F \to k_{o_\lambda}^\times$. We say χ is a finite deformation if $\chi/\bar{\chi}_{\mathrm{lift}}$ is unramified. Here $\bar{\chi}_{\mathrm{lift}}$ is the Teichmller lift of $\bar{\chi}$.*
(3) *Assume that $\ell \neq p$. Let $\rho : G_F \to \mathrm{GL}_2(A)$ be a deformation of $\bar{\rho} : G_F \to \mathrm{GL}_2(k_{o_\lambda})$. We say ρ is a finite deformation if $\det\rho$ is a finite deformation of $\det\bar{\rho}$, and satisfies one of the following conditions:*

- *If $\bar\rho$ is reducible with a twist type $\bar\kappa$, then the I_F-fixed part $(\rho|_{I_F}\otimes\kappa^{-1})^{I_F}$ is A-free and is an A-direct summand, and the rank is equal to $\dim_{k_\lambda}(\bar\rho|_{I_F}\otimes\bar\kappa^{-1})^{I_F}$. Here $\kappa=\bar\kappa_{\mathrm{lift}}: I_F\to W(k_{o_\lambda})^\times\hookrightarrow o_\lambda^\times$ is the Teichmüller lift of $\bar\kappa$.*
- *If $\bar\rho$ is absolutely irreducible and of type 0_{NE}, we only require the condition on the determinant.*
- *If $\bar\rho$ is absolutely irreducible and of type 0_E of the form $\bar\rho=\mathrm{Ind}_{G_{\tilde F}}^{G_F}\bar\psi$, then we require that $\rho|_{I_{\tilde F}}$ is the sum of $\psi|_{I_{\tilde F}}$ and its Frobenius twist. Here $\psi=\bar\psi_{\mathrm{lift}}: G_{\tilde F}\to W(k_{o_\lambda})^\times\hookrightarrow o_\lambda^\times$ is the Teichmüller lift of $\bar\psi$.*

By $F^{\mathbf{u}}_{\bar\rho}:\mathscr{C}^{\mathrm{noeth}}_{o_\lambda}\to\mathbf{Sets}$ (resp. $F^{\mathbf{f}}_{\bar\rho}:\mathscr{C}^{\mathrm{noeth}}_{o_\lambda}\to\mathbf{Sets}$) we denote the functor defined on $\mathscr{C}^{\mathrm{noeth}}_{o_\lambda}$ consisting of the isomorphism classes of the unrestricted deformations (resp. finite deformations). For a continuous character $\chi: G_F\to o_\lambda^\times$, $F^{\mathbf{u}}_{\bar\rho,\chi}$ (resp. $F^{\mathbf{f}}_{\bar\rho,\chi}$) is the subfunctor of $F^{\mathbf{u}}_{\bar\rho}$ (resp. $F^{\mathbf{f}}_{\bar\rho}$) consisting of the deformations whose determinant is χ.

Remark 3.4. *Since we do not assume $\dim_{k_{o_\lambda}}\mathrm{Hom}_{G_F}(\rho,\rho)=1$ in general, these local deformation functors may not be representable, though the restriction to $\mathscr{C}^{\mathrm{artin}}_{o_\lambda}$ always has a versal hull in $\mathscr{C}^{\mathrm{noeth}}_{o_\lambda}$ by [43]. A versal hull represents the deformation functor if $\dim_{k_{o_\lambda}}\mathrm{Hom}_{G_F}(\rho,\rho)=1$.*

3.3. **Tangent spaces outside ℓ.** As usual, we set

$$\mathrm{ad}\,\bar\rho = Hom_{k_{o_\lambda}}(\bar\rho,\ \bar\rho).$$

The tangent space $F^{\mathbf{u}}_{\bar\rho}(k_{o_\lambda}[\epsilon])$ of $F^{\mathbf{u}}_{\bar\rho,}$ is canonically isomorphic to $H^1(F,\ \mathrm{ad}\,\bar\rho)$ as in [32], which is of finite dimension over k_{o_λ} by the finiteness of local Galois cohomology groups. Here $k_{o_\lambda}[\epsilon]$ is the ring of dual numbers. The trace 0-part $\mathrm{ad}^0\,\bar\rho$ of $\mathrm{ad}\,\bar\rho$ gives the tangent space of $F^{\mathbf{u}}_{\bar\rho,\chi}$ consisting of the deformations with the determinant χ. The finite deformations are controlled by the mod ℓ-version of the finite part of Bloch-Kato [1]. For any G_F-module M with order prime to the residual characteristic, we define the finite part by

$$H^1_f(F,\ M)=H^1(F^{\mathrm{unr}}/F,\ M^{I_F})=\ker(H^1(F,M)\longrightarrow H^1(F^{\mathrm{unr}},\ M)).$$

Proposition 3.5. *An element x in $F^{\mathbf{u}}_{\bar\rho}(k_{o_\lambda}[\epsilon])$ corresponds to a finite deformation if and only if it belongs to $H^1_f(F,\ \mathrm{ad}\,\bar\rho)$.*

Proof of Proposition 3.5. In the cases of 2_{PR}, 0_E, and 0_{NE}, this is clear from the definition.

In the case of 1_{PR}, any deformation splits by the vanishing of $H^1(F,\ \bar\mu/\bar\nu)$, so $\rho|_{I_F}$ is a constant deformation.

In the case of 1_{SP}, the I_F-action factors through the maximal pro-ℓ quotient $t_\ell: I_F\twoheadrightarrow\mathbb{Z}_\ell(1)$, so the monodromy group is topologically generated by any element $\sigma\in I_F$ so that $t_\ell(\sigma)$ is a topological generator. We take a basis so that

$$\rho(\sigma)=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

This shows that $\rho|_{I_F}$ is constant. □

The calculation of the dimension of tangent spaces is done as follows. First assume that $\bar\rho$ is absolutely irreducible. Note that the deformation functors are representable in this case.

Lemma 3.6. *Assume that $\bar\rho$ is absolutely irreducible. Then the finite part $H^1_f(F,\ \mathrm{ad}^0\bar\rho)$ always vanishes, and*

$$H^1(F, \mathrm{ad}^0\bar\rho) = \{0\}$$

in the case of 0_{NE}. In the case of 0_E, $H^1(F, \mathrm{ad}^0\bar\rho)$ is one dimensional.

Proof of Lemma 3.6. Since $\bar\rho$ is absolutely irreducible, $H^0(F, \mathrm{ad}^0\bar\rho) = \{0\}$. By the Euler characteristic formula

$$\dim_{k_{o_\lambda}} H^1(F, \mathrm{ad}^0\bar\rho) = \dim_{k_{o_\lambda}} H^0(F, \mathrm{ad}^0\bar\rho) + \dim_{k_{o_\lambda}} H^2(F, \mathrm{ad}^0\bar\rho) = \dim_{k_{o_\lambda}} H^2(F, \mathrm{ad}^0\bar\rho),$$

and $\dim_{k_{o_\lambda}} H^2(F, \mathrm{ad}^0\bar\rho) = \dim_{k_{o_\lambda}} H^0(F, \mathrm{ad}^0\bar\rho(1))$ by the local duality. if $H^0(F, \mathrm{ad}^0\bar\rho(1)) \neq \{0\}$, $\bar\rho \simeq \bar\rho(1)$, and it is easily seen that $\bar\rho$ is of type 0_E and $H^0(F, \mathrm{ad}^0\bar\rho(1))$ is one dimensional. □

3.4. **Local deformation spaces: examples.** Let $\chi : G_F \to o_\lambda^\times$ be an unramified continuous character. In the case of 0_E, $F^{\mathbf{u}}_{\bar\rho,\chi}$ is representable by remark 3.4.

The universal deformation ring R for $F^{\mathbf{u}}_{\bar\rho,\chi}$ is determined in [9], p.141. We recall the result.

$\bar\rho = \mathrm{Ind}^{G_F}_{G_{\tilde F}}\bar\psi$ for the unramified extension $\tilde F$ of F and a character $\bar\psi : G_{\tilde F} \to k^\times_{o_\lambda}$. Take the Teichmüller lift $\psi : G_{\tilde F} \to W(k_{o_\lambda})^\times \hookrightarrow o_\lambda^\times$ of $\bar\psi$. If we consider a deformation $\rho : G_F \to \mathrm{GL}_2(A)$ for a local o_{E_λ}-algebra A fixing the determinant, $\rho|_{G_{\tilde F}}$ decomposes into sum of different characters, and for a suitable lift ϕ of $\bar\psi$, $\rho = \mathrm{Ind}^{G_F}_{G\tilde F}\phi$. $\xi = \phi/\psi$ is a character of $G_{\tilde F}$ of ℓ-power order. By the local class field theory ξ is a character of $o^\times_{\tilde F}$ of ℓ-power order.

By the consideration,

Proposition 3.7 (Diamond)**.** *R is isomorphic to the group ring $o_{E_\lambda}[\Delta_{\tilde F}]$. Here $\Delta_{\tilde F}$ is the ℓ-Sylow subgroup of $k^\times_{\tilde F}$.*

We need one more case where a versal hull is explicitly determined.

Proposition 3.8 (Faltings)**.** *Let $\bar\rho$ be an unramified representation with different Frobenius eigenvalues, and $q \equiv 1 \mod \ell$. Then the versal hull of $F^{\mathbf{u}}_{\bar\rho,\chi}$ over o_λ fixing the determinant is isomorphic to $o_\lambda[[\widehat{(F^\times)}_\ell]]$. Here $\widehat{(F^\times)}_\ell$ is the pro-ℓ-completion of $F^\times$.*

This is found in [49], Appendix, p. 569. In fact, one shows that any lifting ρ to an artinian local o_λ-algebra A is decomposable under the assumptions of 3.8. We construct an isomorphism explicitly for our later needs.

We choose a decomposition $\bar\rho = \bar\chi_1 \oplus \bar\chi_2$. $\chi_1,\ \chi_2$ denote the Teichmüller liftings of $\bar\chi_1,\ \bar\chi_2$. We may assume that ρ takes the form

$$\rho = \begin{pmatrix} \tilde\chi_1 & 0 \\ 0 & \tilde\chi_2 \end{pmatrix}$$

for liftings $\tilde\chi_1,\ \tilde\chi_2$ of $\bar\chi_1,\ \bar\chi_2$.

$\delta = \tilde\chi_2/\chi_2 : G_F \to A^\times$ is a character of ℓ-power order. So it factors through $G_F \overset{\delta^{\mathrm{univ}}}{\to} o_\lambda[[\widehat{(F^\times)}_\ell]] \to A$ by the local class field theory. Note that $\widehat{(F^\times)}_\ell$ is isomorphic to $\mathbb{Z}_\ell \times \Delta_F$ if we choose a uniformizer of F. Here Δ_F is the ℓ-Sylow subgroup of $k^\times_F$.

In these examples, the local deformation space is the group ring of a commutative group.

3.5. **Deformations at ℓ: nearly ordinary and flat deformations.** In this subsection, we assume that the residual characteristic p of o_F is equal to ℓ.

Definition 3.9. *Let A be a complete local noetherian ring with the maximal ideal m_A and the finite residue field k_A of characteristic ℓ.*

(1) *A pair (ρ, μ) of a continuous G_F-representation $\rho : G_F \to \mathrm{GL}_2(A)$ and a continuous character $\mu : G_F \to A^\times$ is called nearly ordinary if ρ is isomorphic to*

$$\rho \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix},$$

where $\chi_1 = \mu$. μ is called the nearly ordinary character for (ρ, μ), and the restriction $\kappa = \mu|_{I_F}$ is called the nearly ordinary type of (ρ, μ).

(2) *When A is a field, a reducible representation $\rho : G_F \to \mathrm{GL}_2(A)$ is called G_F-distinguished if one of the following conditions holds. ρ is indecomposable, or it is semi-simple and has two distinct constituants.*

By the local class field theory, $(I_F^{\mathrm{ab}})_{G_F} \simeq o_F^\times$, so a nearly ordinary type is regarded as a character of $o_F^\times$.

When a nearly ordinary character is clearly specified, we say ρ is a nearly ordinary representation with the nearly ordinary character μ for short.

Definition 3.10. *For any complete local noetherian ring A with the maximal ideal m_A and the finite residue k_A of characteristic ℓ,*

(1) *We define*

$$H^1_{\mathrm{fl}}(o_{F^{\mathrm{unr}}},\ A(1)) \overset{def}{=} \varprojlim_n H^1_{\mathrm{fl}}(o_{F^{\mathrm{unr}}},\ A/m_A^n(1)) = \varprojlim_n o_{F^{\mathrm{unr}}}^\times/(o_{F^{\mathrm{unr}}}^\times)^{\ell^n} \otimes_{\mathbb{Z}_\ell} A,$$

$$H^1(F^{\mathrm{unr}},\ A(1)) \overset{def}{=} \varprojlim_n H^1(F^{\mathrm{unr}},\ A/m_A^n(1)) = \varprojlim_n (F^{\mathrm{unr}})^\times/((F^{\mathrm{unr}})^\times)^{\ell^n} \otimes_{\mathbb{Z}_\ell} A.$$

(2) *The exact sequence*

$$1 \longrightarrow o_{F^{\mathrm{unr}}}^\times \longrightarrow (F^{\mathrm{unr}})^\times \longrightarrow \mathbb{Z} \longrightarrow 0$$

induces

$$0 \longrightarrow H^1_{\mathrm{fl}}(o_{F^{\mathrm{unr}}},\ A(1)) \longrightarrow H^1(F^{\mathrm{unr}},\ A(1))) \xrightarrow{\mathrm{ev}_A} A \longrightarrow 0.$$

We call ev_A the evaluation map.

$H^1_{\mathrm{fl}}(o_{F^{\mathrm{unr}}},\ A(1))$ (resp. $H^1(F^{\mathrm{unr}},\ A(1))$ classifies (pro-systems of) the extensions $0 \to A \otimes_{\mathbb{Z}_\ell} \varprojlim_n \mu_{\ell^n} \to M \to A \to 0$ over $o_{F^{\mathrm{unr}}}$ (resp. F^{unr}) with the flat (resp. étale) topology.

Definition 3.11. *Let A be a complete local noetherian ring with the maximal ideal m_A and the finite residue k_A of characteristic ℓ, $\rho : G_F \to \mathrm{GL}_2(A)$ a nearly ordinary representation of the form*

$$0 \longrightarrow \chi_1 \longrightarrow \rho \longrightarrow \chi_2 \longrightarrow 0,$$

where $\chi_1 : G_F \to A^\times$ is the nearly ordinary character.

(1) *Assume that ρ satisfies $\chi_1|_{I_F} = \chi_2(1)|_{I_F}$. Then c_ρ is the extension class in $H^1(F^{\mathrm{unr}},\ A(1))$ defined by $\rho \otimes \chi_2^{-1}$.*

(2) *A nearly ordinary representation ρ is called nearly ordinary finite if $\chi_1|_{I_F} = \chi_2(1)|_{I_F}$, and if the class c_ρ belongs to $H^1_{\mathrm{fl}}(o_{F^{\mathrm{unr}}},\ A(1)) \subset H^1(F^{\mathrm{unr}},\ A(1))$. Equivalently, $\mathrm{ev}_A(c_\rho) = 0$.*

Proposition 3.12. *Let E_λ be an ℓ-adic field, A an o_{E_λ}-finite flat local algebra, $\rho : G_F \to \mathrm{GL}_2(A)$ a nearly ordinary representation with the nearly ordinary character χ which satisfies $\det\rho|_{I_F} = (\chi|_{I_F})^2(-1)$. Assume that there is an ℓ-divisible group M over o_F whose Tate module $T_\ell(M_F)$ satisfies*

$$T_\ell(M_F) \otimes_{o_{E_\lambda}} E_\lambda \simeq (\rho \otimes \chi^{-1}(1)) \otimes_{o_{E_\lambda}} E_\lambda.$$

Then ρ is a nearly ordinary finite representation.

Proof of Proposition 3.12. We may assume that $\chi = 1$ by a twisting. Let c_ρ be the extension class in $H^1(F^{\mathrm{unr}},\ A(1))$ which corresponds to $\rho|_{F^{\mathrm{unr}}}$. It suffices to prove $\mathrm{ev}_A(c_\rho) = 0$ for the evaluation map ev_A in Definition 3.10 (2).

By the nearly ordinariness and the assumption on the determinant of ρ, the generic fiber M_F admits a filtration by an ℓ-divisible subgroup over F

$$0 \longrightarrow M_{1,F} \longrightarrow M_F \to M_{2,F} \longrightarrow 0,$$

where $T_\ell(M_{i,F})$ are $A \otimes_{o_{E_\lambda}} E_\lambda$-modules of rank one, and both $T_\ell(M_{1,F}(-1))$ and $T_\ell(M_{2,F})$ are unramified as Galois modules. We view $M_{1,F}$ as the generic fiber of an ℓ-divisible group of multiplicative type M_1. By [45], Proposition 12, there is a morphism $i : M_1 \to M$ as ℓ-divisible groups which gives the exact sequence over F as above. This implies that the multiplicative part M^{mult} of M satisfies $T_\ell(M^{\mathrm{mult}})_F \simeq T_\ell(M_1)_F$. Thus we may assume that i is a closed immersion as an ℓ-divisible group, and the quotient M_2 is étale.

The exact sequence of ℓ-divisible groups over $o_{F^{\mathrm{unr}}}$

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

defines, in the fppf-topology, an element c_M of $\varprojlim_n H^1_{\mathrm{fl}}(o_{F^{\mathrm{unr}}},\ \mathrm{Hom}_A(M_2[\ell^n], M_1[\ell^n])) = H^1_{\mathrm{fl}}(o_{F^{\mathrm{unr}}},\ \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} N$. Here $N = \mathrm{Hom}_A(T_\ell(M_2), T_\ell(M_1(-1)))$. On the other hand, since for the rank 2-lattice $L = A^{\oplus 2}$, the representation $\rho(1)$ is defined, and the subspace L_1 where ρ acts by $\chi_1(1)$ satisfies $L_1 \otimes_{o_{E_\lambda}} E_\lambda = M_1 \otimes_{o_{E_\lambda}} E_\lambda$. By this identification,

$$(H^1(F^{\mathrm{unr}},\ \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} N) \otimes_{o_{E_\lambda}} E_\lambda = H^1(F^{\mathrm{unr}}, A(1)) \otimes_{o_{E_\lambda}} E_\lambda,$$

and

$$(H^1_{\mathrm{fl}}(o_{F^{\mathrm{unr}}},\ \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} N) \otimes_{o_{E_\lambda}} E_\lambda = H^1_{\mathrm{fl}}(o_{F^{\mathrm{unr}}}, A(1)) \otimes_{o_{E_\lambda}} E_\lambda$$

hold, and c_ρ spans the same E_λ-subspace as c_M in $H^1(F^{\mathrm{unr}}, A(1)) \otimes_{o_{E_\lambda}} E_\lambda$. This implies that $\mathrm{ev}_A(c_\rho)$ is zero in $A \otimes_{o_{E_\lambda}} E_\lambda$ because c_M belongs to $H^1_{\mathrm{fl}}(o_{F^{\mathrm{unr}}},\ \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} N$. Since A is ℓ-torsion free, $\mathrm{ev}_A(c_\rho)$ is zero in A, and hence ρ is nearly ordinary finite. □

Now we introduce deformation functors for nearly ordinary representations.

Definition 3.13. *Let o_λ be a complete local noetherian ring with the maximal ideal m_{o_λ} and the finite residue field k_{o_λ} of characteristic ℓ. We fix continuous characters $\kappa : (I_F^{\mathrm{ab}})_{G_F} \to o_\lambda^\times$, and $\chi : G_F \to o_\lambda^\times$. Let $\bar\rho : G_F \to \mathrm{GL}_2(k_{o_\lambda})$ be a G_F-distinguished nearly ordinary representation whose nearly ordinary type $\bar\kappa$ is $\kappa \mod m_{o_\lambda}$, and the determinant is $\chi \mod m_{o_\lambda}$.*

(1) *We define the functor*

$$F^{\mathbf{n.o.}}_{\bar\rho,\kappa} : \mathscr{C}^{\mathrm{noeth}}_{o_\lambda} \to \mathbf{Sets}$$

as follows: for each $A \in \mathrm{ob}\mathscr{C}^{\mathrm{noeth}}_{o_\lambda}$, $F^{\mathbf{n.o.}}_{\bar\rho,\kappa}(A)$ is the set consisting of isomorphism classes of (ρ, ξ). Here $\rho : G_F \to \mathrm{GL}_2(A)$ is a nearly ordinary representation with the nearly ordinary type κ, $\xi : (\rho \mod m_A) \simeq \bar\rho$ is an isomorphism as a G_F-representation.

$F^{\mathbf{n.o.}}_{\bar\rho,\kappa,\chi}$ denotes the subfunctor of $F^{\mathbf{n.o.}}_{\bar\rho,\kappa}$ consisting of the deformations whose determinant is χ.

(2) *Assume moreover that* $\bar{\rho}|_{I_F}$ *has a form*

$$0 \longrightarrow \bar{\kappa} \longrightarrow \bar{\rho} \longrightarrow \bar{\kappa}(-1) \longrightarrow 0.$$

Then

$$F^{\mathbf{n.o.f}}_{\bar{\rho},\kappa} : \mathscr{C}^{\mathrm{noeth}}_{o_\lambda} \longrightarrow \mathbf{Sets}$$

is the subfunctor of $F^{\mathbf{n.o.}}_{\bar{\rho},\kappa}$ *consisting of the nearly ordinary finite deformations.* $F^{\mathbf{n.o.f}}_{\bar{\rho},\kappa,\chi}$ *denotes the subfunctor of* $F^{\mathbf{n.o.}}_{\bar{\rho},\kappa}$ *consisting of the deformations whose determinant is* χ*.*

Let F be a local field of mixed characteristic p. We assume that F is absolutely unramified, and $p \geq 3$. We recall basic results on finite flat commutative group schemes over o_F.

For a noetherian scheme S, let $\mathrm{GSch}^{\mathrm{ff}}{}_S$ be the category of commutative finite flat group schemes over S.

By Raynaud's result [36], $\mathrm{GSch}^{\mathrm{ff}}{}_{o_F}$ is an abelian category, and the restriction functor

$$\mathrm{Res}^{\mathrm{GSch}^{\mathrm{ff}}}|_F : \mathrm{GSch}^{\mathrm{ff}}{}_{o_F} \longrightarrow \mathrm{GSch}^{\mathrm{ff}}{}_F$$

is fully-faithful. Since the charateristic of F is zero, all finite flat group schemes over F are étale, and hence are regarded as a commutative finite group with a G_F-action. For $G \in \mathrm{GSch}^{\mathrm{ff}}{}_{o_F}$, we define the associated G_F-module $DT(G)$ by

$$DT(G) = \mathrm{Hom}(G(\bar{F}), \mathbb{Q}/\mathbb{Z}).$$

DT is a cofunctor from $\mathrm{GSch}^{\mathrm{ff}}{}_{o_F}$ to the category of G_F-modules $\mathscr{G}_F$, which is fully faithful. The essential image of DT is denoted by $\mathscr{G}_{[0,1],F}$.

For a commutative ring A, an A-action on a finite flat group scheme G over S is a ring homomorphism $A \to \mathrm{End}_S G$. By $\mathrm{GSch}^{\mathrm{ff}}{}_{A,S}$, we denote the category of commutative finite flat group schemes with an A-action over S.

Let $\mathscr{G}^A_{[0,1],F}$ be the subcategory of $\mathscr{G}_{[0,1],F}$ consisting of G_F-modules with an A-aciton. $\mathscr{G}^A_{[0,1],F}$ is equivalent to $\mathrm{GSch}^{\mathrm{ff}}{}_{A,o_F}$.

Assume that A is a commutative ring of characteristic p. The category $M^{FL}_{[0,1],A}$ is defined as follows. $M^{FL}_{[0,1],A}$ consists of triples $D = (F^0(D), F^1(D), \varphi)$ such that

- $F^0(D)$ is a finite $o_F \otimes_{\mathbb{Z}_p} A$-module.
- $F^1(D)$ is an $o_F \otimes_{\mathbb{Z}_p} A$-submodule of $F^0(D)$.
- φ is a σ-linear A-isomorphism $F^1(D) \oplus (F^0(D)/F^1(D)) \simeq F^0(D)$. Here $\sigma : o_F \simeq o_F$ is the lift of the absolute frobenius of k_F.

By [14], there is an equivalence of categories

$$D : \mathscr{G}^A_{[0,1],F} \simeq M^{FL}_{[0,1],A}.$$

A quasi-inverse functor of D is denoted by V.

Definition 3.14. *We assume that* F *is absolutely unramified local field of the residual characteristic* p*,* $p \geq 3$*. Let* A *be a complete local noetherian* o_λ*-algebra with a finite residue field of characteristic* p*.*

(1) *A continuous representation* $\rho : G_F \to \mathrm{GL}_2(A)$ *is called flat if there exists a continuous character* $\mu : G_F \to A^\times$ *satisfying the following conditions.*
   - *For any integer* $n \geq 1$*,* $(\rho \otimes \mu^{-1}) \mod m_A^n$ *belongs to* $\mathscr{G}_{F,[0,1]}$*.*
   - $\det(\rho \otimes \mu^{-1})|_{I_F} = \chi^{-1}_{\mathrm{cycle}}|_{I_F}$*.*

   We call μ *a twist character of* ρ*, and* $\mu|_{I_F}$ *a twist type of* ρ*.*

(2) *Let* $\bar{\rho} : G_F \to \mathrm{GL}_2(k_{o_\lambda})$ *be a* G_F*-representation,* $\kappa : (I_F)_{G_F} \to o_\lambda^\times$ *a continuous character.*

$$F^{\mathbf{fl}}_{\bar{\rho},\kappa} : \mathscr{C}^{\mathrm{noeth}}_{o_\lambda} \longrightarrow \mathbf{Sets}$$

is the functor defined as follows: for each $A \in \mathrm{ob}\mathscr{C}^{\mathrm{noeth}}_{o_\lambda}$, $F^{\mathbf{fl}}_{\bar{\rho},\kappa}(A)$ *is the set of isomorphism classes of flat representations with the twist type* κ*, equipped with an isomorphism* $\rho \mod m_A \simeq \bar{\rho}$*. For a continuous character* $\chi : G_F \to o_\lambda^\times$, $F^{\mathbf{fl}}_{\bar{\rho},\kappa,\chi}$ *is the subfunctor of* $F^{\mathbf{fl}}_{\bar{\rho},\kappa}$ *consisting of the deformations with the determinant* χ*.*

Remark 3.15. (1) *Assume that* F *is absolutely unramified with the residual characteristic* $p \geq 3$*. A nearly ordinary finite representation with a nearly ordinary type* κ *is a flat representation with a twist type* κ*.*
(2) *For flat deformations, we allow the case when* $\bar{\rho}$ *is absolutely reducible (*$\bar{\rho}$ *can be split). This becomes especially important when* $F \neq \mathbb{Q}_p$*.*

3.6. **Tangent spaces at** ℓ**.** We discuss the nearly ordinary case first. Since we allow ramifications of F, the tangent space calculation is discussed in detail.

We fix characters $\kappa : (I_F^{\mathrm{ab}})_{G_F} \to o_\lambda^\times$, and $\chi : G_F \to o_\lambda^\times$. Let $\bar{\rho} : G_F \to \mathrm{GL}_2(k_{o_\lambda})$ be a nearly ordinary representation whose nearly ordinary type $\bar{\kappa}$ and $\det\bar{\rho}$ are equal to $\kappa \mod m_{o_\lambda}$ and $\chi \mod m_{o_\lambda}$ respectively. Let $G = \mathrm{GL}_{2,k_\lambda}$ as an algebraic group over k_λ, Z the center of G, B the standard Borel subgroup consisting of the upper triangular matrices in G, U the uniportent radical of B, $T = B/U$. By $k_{o_\lambda}[\epsilon]$ we mean the ring of dual numbers.

Let $\rho : G_F \to G(k_{o_\lambda}[\epsilon])$ be a deformation of $\bar{\rho}$ whose nearly ordinary type and the determinant are fixed to κ and χ. Then ρ defines the Kodaira-Spencer class $c_G(\rho)$ in $H^1(F, \mathrm{ad}^0\,\bar{\rho})$. If we regard G-representation $\mathrm{ad}_{G/Z}$ as a G_F-representation by ρ, it is identified with $\mathrm{ad}^0\,\bar{\rho}$ (projective deformations are the same as the determinant fixed deformations if $\ell \neq 2$).

Since $\bar{\rho}$ is nearly ordinary, we may assume that $\bar{\rho}$ factors through $B(k_{o_\lambda}) \hookrightarrow G(k_{o_\lambda})$ by taking a suitable basis. We consider the filtration as a B-representation $0 \subset \mathrm{ad}_{B/Z} \subset \mathrm{ad}_{G/Z}$, where we identify $\mathrm{ad}_{B/Z} \subset \mathrm{ad}_{G/Z}$ with $W_1 = \bar{\rho} \otimes (\bar{\chi}_2)^{-1} \subset W = \mathrm{ad}^0\,\bar{\rho}$ as G_F-representations.

Since $\rho : G_F \to G(k_{o_\lambda}[\epsilon])$ is a nearly ordinary deformation of $\bar{\rho}$, by a suitable choice of a basis, ρ takes the values in $B(k_{o_\lambda}[\epsilon])$, and hence the cohomology class $c_G(\rho)$ in $H^1(F, \mathrm{ad}^0\,\bar{\rho})$ associated to ρ is the image of the class $c_B(\rho)$ in $H^1(F, \mathrm{ad}_{B/Z})$, defined similarly as $c_G(\rho)$ as the Kodaira-Spencer class.

$c_B(\rho)$ must satisfy one more condition since we fix the nearly ordinary type. $\mathrm{ad}_{B/Z}$ admits a filtration $\mathrm{ad}_U \subset \mathrm{ad}_{B/Z}$ such that we have the exact sequence

$$0 \longrightarrow \mathrm{ad}_U \longrightarrow \mathrm{ad}_{B/Z} \longrightarrow \mathrm{ad}_{T/Z} \longrightarrow 0$$

of B-representations. It induces the exact sequence

$$0 \longrightarrow \bar{\chi}_1/\bar{\chi}_2 \longrightarrow W_1 \longrightarrow k_{o_\lambda} \to 0$$

as G_F-representations. Since we are considering deformations with a fixed nearly ordinary type, the deformations in the direction of T/Z should be unramified, and hence the image of $c_B(\rho)$ in $H^1(F, k_{o_\lambda})$ belongs to $H^1(F^{\mathrm{unr}}/F, k_{o_\lambda})$.

Theorem 3.16. *Let* $\bar{\rho}$ *be a* G_F*-distinguished nearly ordinary representation.*

(1) *The dimension of* $F^{\mathbf{n.o.}}_{\bar{\rho},\kappa,\chi}(k_{o_\lambda}[\epsilon])$ *is at most* $\dim_{k_{o_\lambda}} H^1(F,\ \bar{\chi}_1/\bar{\chi}_2) + \dim_{k_{o_\lambda}} H^0(F,\ \mathrm{ad}^0\,\bar{\rho}) - \dim_{k_{o_\lambda}} H^0(F,\ \bar{\chi}_1/\bar{\chi}_2)$*. (This upper bound is equal to* $\dim_{k_{o_\lambda}} H^0(F,\ \mathrm{ad}^0\,\bar{\rho}) + [F : \mathbb{Q}_p] + \dim_{k_{o_\lambda}} H^0(F,\ (\bar{\chi}_2/\bar{\chi}_1)(1))$ *by the Euler characteristic formula of Tate).*
(2) *If* $\bar{\chi}_2 = \bar{\chi}_1(-1)$*, and* $\bar{\rho}$ *is not nearly ordinary finite, then the dimension of* $F^{\mathbf{n.o.}}_{\bar{\rho},\kappa,\chi}(k_{o_\lambda}[\epsilon])$ *is at most* $\dim_{k_{o_\lambda}} H^0(F, \mathrm{ad}^0\,\bar{\rho}) + [F : \mathbb{Q}_p]$*.*

(3) *If* $\bar{\chi}_2 = \bar{\chi}_1(-1)$, *and* $\bar{\rho}$ *is nearly ordinary finite, then the dimension of* $F^{\mathbf{n.o.f.}}_{\bar{\rho},\kappa,\chi}(k_{o_\lambda}[\epsilon])$ *is at most* $\dim_{k_{o_\lambda}} H^0(F, \mathrm{ad}^0 \bar{\rho}) + [F : \mathbb{Q}_p]$.

Before beginning the proof of 3.16, we prepare several lemmas.

Lemma 3.17. *Let* $\bar{\rho}$ *be a* G_F*-distinguished nearly ordinary representation.*

(1) *If* $\bar{\rho}$ *is semi-simple, then* $\dim_{k_{o_\lambda}} H^0(F, \mathrm{ad}^0 \bar{\rho}) = \dim_{k_{o_\lambda}} H^0(F, W_1) = 1 + \dim_{k_{o_\lambda}} H^0(F, \bar{\chi}_1/\bar{\chi}_2) = 1$, *and* $H^1(F, \bar{\chi}_1/\bar{\chi}_2) \to H^1(F, W_1)$ *is injective.*

(2) *If* $\bar{\rho}$ *is not semi-simple, then* $\dim_{k_{o_\lambda}} H^0(F, \mathrm{ad}^0 \bar{\rho}) = \dim_{k_{o_\lambda}} H^0(F, W_1) = \dim_{k_{o_\lambda}} H^0(F, \bar{\chi}_1/\bar{\chi}_2)$, *and the kernel* $L_{\bar{\rho}}$ *of* $H^1(F, \bar{\chi}_1/\bar{\chi}_2) \to H^1(F, W_1)$ *is one dimensional.*

The verification of this lemma is easy, and left to the reader.

Lemma 3.18. *Assume that* $\ell \geq 3$, $\bar{\chi}_2 = \bar{\chi}_1(-1)$, *and* $\bar{\rho}$ *is not nearly ordinary finite. For any nearly ordinary deformation* ρ *over* $k_{o_\lambda}[\epsilon]$ *with a fixed nearly ordinary character and determinant of the form*

$$0 \longrightarrow \tilde{\chi}_1 \longrightarrow \rho \longrightarrow \tilde{\chi}_2 \longrightarrow 0,$$

$\tilde{\chi}_2 = \tilde{\chi}_1(-1)$ *holds.*

Proof of Lemma 3.18. When we view $\bar{\chi}_1$ and $\tilde{\chi}_2$ as characters of G_F with the values in $k_{o_\lambda}[\epsilon]$ by $k_{o_\lambda} \hookrightarrow k_{o_\lambda}[\epsilon]$, we denote them by χ_1 and χ_2.

$\mu = \tilde{\chi}_1/\chi_1 : G_F \to k_{o_\lambda}[\epsilon]^\times$ is an unramified character lifting the trivial character. $\tilde{\chi}_2 = \chi_2 \cdot (\mu)^{-1}$ since we fix the determinant. $\rho \otimes (\tilde{\chi}_2)^{-1}$ defines an element $c(\rho) \in \mathrm{Ext}^1(F,\ \tilde{\chi}_1/\tilde{\chi}_2^{-1}) = H^1(F,\ \mu^2\chi_1/\chi_2) = H^1(F, \mu^2(1))$ which lifts the class $c(\bar{\rho})$ in $H^1(F,\ \bar{\chi}_1/\bar{\chi}_2) = H^1(F, k_{o_\lambda}(1))$ defined by $\bar{\rho} \otimes \bar{\chi}_2^{-1}$ by the canonical homomorphism

$$H^1(F,\ \mu^2(1)) \longrightarrow H^1(F,\ k_{o_\lambda}(1))$$

deduced from

$$\mu^2(1) \longrightarrow k_{o_\lambda}(1) \longrightarrow 0.$$

$c(\bar{\rho})|_{F^{\mathrm{unr}}} = c_{\bar{\rho}}$ in $H^1(F^{\mathrm{unr}},\ k_{o_\lambda}(1))$.

Assume that $\mu \neq 1$. Consider the following homomorphism

$$H^1(F^{\mathrm{unr}}, \mu^2(1)) \simeq (F^{\mathrm{unr}})^\times/((F^{\mathrm{unr}})^\times)^\ell \otimes_{\mathbb{F}_\ell} \mu^2 \xrightarrow{\tilde{\beta}} \mu^2,$$

where $\tilde{\beta}$ is deduced from the valuation $(F^{\mathrm{unr}})^\times \to \mathbb{Z}$ (evaluation map, see Definition 3.10).

$\tilde{\beta}(c(\rho)|_{F^{\mathrm{unr}}})$ is fixed by $\mathrm{Gal}(F^{\mathrm{unr}}/F)$. Since $\ell \neq 2$, $\mu^2 \neq 1$, and $(\mu^2)^{\mathrm{Gal}(F^{\mathrm{unr}}/F)}$ is the subspace $k_{o_\lambda}\epsilon$ spanned by ϵ. Then $\tilde{\beta}(c(\rho)|_{F^{\mathrm{unr}}})$ is mapped to zero by $\mu^2 \to \mu^2 \otimes_{k_{o_\lambda}[\epsilon]} k_{o_\lambda} = k_{o_\lambda}$, which is equal to $\beta(c_{\bar{\rho}})$ where $\beta = \mathrm{ev}_{k_{o_\lambda}} : H^1(F^{\mathrm{unr}}, k_{o_\lambda}(1)) \simeq (F^{\mathrm{unr}})^\times/((F^{\mathrm{unr}})^\times)^\ell \otimes_{\mathbb{F}_\ell} k_{o_\lambda} \xrightarrow{\mathrm{ev}_{k_{o_\lambda}}} k_{o_\lambda}$ is the evaluation map for k_{o_λ} (Definition 3.10). This implies that $\bar{\rho}$ is nearly ordinary finite. $\square$

Proof of Theorem 3.16. For (1), assume that $\bar{\rho}$ is semi-simple. Then $H^1(F, \bar{\chi}_1/\bar{\chi}_2) \to H^1(F, W_1)$ is injective by Lemma 3.17, (1). So the possible maximal dimension is $\dim_{k_{o_\lambda}} H^1(F, \bar{\chi}_1/\bar{\chi}_2) + 1$. If $\bar{\rho}$ is not semi-simple, the kernel $L_{\bar{\rho}}$ of $H^1(F, \bar{\chi}_1/\bar{\chi}_2) \to H^1(F, W_1)$ is one dimensional (this is the subspace spanned by the extension determined by $\bar{\rho}$). So the dimension is at most $\dim_{k_{o_\lambda}} H^1(F, \bar{\chi}_1/\bar{\chi}_2)$.

For (2), by Lemma 3.18, there are no deformations in the direction of $\mathrm{ad}_{T/Z}$, and the possible maximal dimension is $\dim_{k_{o_\lambda}} H^1(F, \bar{\chi}_1/\bar{\chi}_2)/L_{\bar{\rho}} = \dim_{k_{o_\lambda}} H^1(F, \bar{\chi}_1/\bar{\chi}_2) - 1$, which is equal to $[F : \mathbb{Q}_p] + \dim_{k_{o_\lambda}} H^0(F, \bar{\chi}_1/\bar{\chi}_2) = [F : \mathbb{Q}_p] + \dim_{k_{o_\lambda}} H^0(F, \mathrm{ad}^0 \bar{\rho})$ by Lemma 3.17 and the Euler characteristic formula.

For (3), first assume that $\bar{\rho}$ is semi-simple. For a class c which belongs to the tangent space, the set of the classes in the tangent space with the same projection to $H^1(F, k_{o_\lambda})$ is a torsor under $H^1_{\mathrm{fl}}(o_F, k_{o_\lambda}(1)) \simeq o_F^\times/(o_F^\times)^\ell \otimes_{\mathbb{F}_\ell} k_{o_\lambda}$. So the dimension is estimated from above by $1 + [F : \mathbb{Q}_p] + \dim_{\mathbb{F}_\ell} H^0(F, \mu_\ell)$. Since we assume that $\bar{\rho}$ is G_F-distinguished, $H^0(F, \mu_\ell) = \{1\}$, and the result follows.

If $\bar{\rho}$ is not semi-simple, the kernel $L_{\bar{\rho}}$ of $H^1(F, \bar{\chi}_1/\bar{\chi}_2) \to H^1(F, W_1)$ belongs to $H^1_{\mathrm{fl}}(o_F, \mu_\ell \otimes_{\mathbb{F}_\ell} k_{o_\lambda})$ since $\bar{\rho}$ is nearly ordinary finite.

So the dimension of the tangent space is estimated from above by $\dim_{k_{o_\lambda}} H^1_{\mathrm{fl}}(o_F, \mu_\ell \otimes_{\mathbb{F}_\ell} k_{o_\lambda})/L_{\bar{\rho}} + 1 = (\dim_{\mathbb{F}_\ell} H^1_{\mathrm{fl}}(o_F, \mu_\ell) - 1) + 1 = [F : \mathbb{Q}_p] + \dim_{\mathbb{F}_\ell} H^0(F, \mu_\ell) = [F : \mathbb{Q}_p] + \dim_{k_{o_\lambda}} H^0(F, \mathrm{ad}^0\,\bar{\rho})$ by Lemma 3.17, (2). □

Remark 3.19. *In the terminology of* [50]*, in the case of 3.16 (2), Selmer deformations are strict by a theorem of Diamond, and the dimension estimate follows from proposition 1.9, (iv).*

Next we consider flat deformations. The case was treated by Ramakrishna [37] when $F = \mathbb{Q}_p$ and $\bar{\rho}$ is absolutely irreducible. Conrad discusses flat deformations in detail for $F = \mathbb{Q}_p$ ([4], theorem 5.1), though he excludes the case when $\bar{\rho}$ is split.

Here we only discuss the tangent space of flat deformation functors by a use of the theory of Fontaine-Laffaille (cf. [37], [49]).

Theorem 3.20. *Assume that F is absolutely unramified, with the residual characteristic $p \geq 3$. Let $\bar{\rho}$ be a flat representation with the twist character $\bar{\kappa}$, $\kappa : (I_F)_{G_F} \to o_\lambda^\times$ (resp. $\chi : (I_F)_{G_F} \to o_\lambda^\times$) a continuous character lifting $\bar{\kappa}$ (resp.* $\det\bar{\rho}|_{I_F}$*).*

1. *The dimension of $F^{\mathbf{fl}}_{\bar{\rho},\kappa}(k_{o_\lambda}[\epsilon])$ is equal to* $\dim_{k_{o_\lambda}} H^0(F,\ \mathrm{ad}\,\bar{\rho}) + [F : \mathbb{Q}_p]$.
2. *The dimension of $F^{\mathbf{fl}}_{\bar{\rho},\kappa,\chi}(k_{o_\lambda}[\epsilon])$ is equal to* $\dim_{k_{o_\lambda}} H^0(F,\ \mathrm{ad}^0\,\bar{\rho}) + [F : \mathbb{Q}_p]$.

Lemma 3.21. *Let $\rho : G_F \to \mathrm{GL}_2(A)$ be a flat deformation of $\bar{\rho} : G_F \to \mathrm{GL}_2(k)$ with the trivial twist type, where A is an object of $\mathscr{C}^{\mathrm{artin}}_{o_\lambda}$ of characteristic p. Then $F^0(D(\rho))$ is free of rank 2 over $o_F \otimes_{\mathbb{Z}_p} A$, and $F^1(D(\rho))$ is free of rank 1 over $o_F \otimes_{\mathbb{Z}_p} A$.*

Proof of Lemma 3.21. First we treat the case of $A = k$. Assume that $\bar{\rho}$ is absolutely reducible. By extending k if necessary, we may assume that $\bar{\rho}$ is reducible of the form

$$0 \longrightarrow \bar{\chi}_1 \longrightarrow \bar{\rho} \longrightarrow \bar{\chi}_2 \longrightarrow 0,$$

where $\bar{\chi}_1$ and $\bar{\chi}_2$ are characters, and there is an embedding $k_F \hookrightarrow k$. For $i = 1, 2$, $\bar{\chi}_i|_{I_F}$ is expressed as

$$\bar{\chi}_i|_{I_F} = \prod_{\sigma \in \mathrm{Gal}(F/\mathbb{Q}_p)} (\chi_F^\sigma)^{-\epsilon_{i,\sigma}},$$

where $\chi_F : I_F \to (I_F^{\mathrm{ab}})_{G_F} \to k_F^\times \hookrightarrow k^\times$ is the fundamental character of F, and χ_F^σ denotes the σ-twist of χ_F. Since $\bar{\chi}_i$, $i = 1, 2$, is associated with a finite flat group scheme, $\epsilon_{i,\sigma} \in \{0, 1\}$ for any $\sigma \in \mathrm{Gal}(F/\mathbb{Q}_p)$.

By our assumption, $\det\bar{\rho}|_{I_F} = \bar{\chi}_{\mathrm{cycle}}^{-1}|_{I_F}$, and hence we have an equality

$$(*) \qquad \epsilon_{1,\sigma} + \epsilon_{2,\sigma} = 1.$$

for $\sigma \in \mathrm{Gal}(F/\mathbb{Q}_p)$.

The exact sequence

$$0 \longrightarrow D(\bar{\chi}_1) \longrightarrow D(\bar{\rho}) \longrightarrow D(\bar{\chi}_2) \longrightarrow 0$$

is strictly compatible with the filtrations. $F^0(D(\bar{\chi}_i))$ for $i = 1, 2$ are free of rank 1 over $k_F \otimes k$, and hence $F^0(D(\bar{\rho}))$ is free of rank 2. For $i = 1, 2$, $F^1(D(\bar{\chi}_i))$ is the part of $F^0(D(\bar{\chi}_i))$ where $k_F^\times$ acts as $\prod_{\sigma \in \mathrm{Gal}(F/\mathbb{Q}_p)} (\iota_\sigma)^{\epsilon_{i,\sigma}}$. Here $\iota_\sigma : k_F^\times \xrightarrow{\sigma} k_F^\times \hookrightarrow k^\times$. By $(*)$, $F^1(D(\bar{\rho}))$ is free of rank 1 over $k_F \otimes k$.

In the absolutely irreducible case, let $\tilde{F}$ be the unramified extension of F of degree 2. To show 3.21 for $\bar{\rho}$, it suffices to prove it for $\bar{\rho}|_{\tilde{F}}$. Since $\bar{\rho}|_{\tilde{F}}$ is absolutely reducible, we are reduced to the absolutely reducible case, which is already treated.

In the general case, we proceed by an induction on $n = \mathrm{length}A$. For $n = 1$, the claim is already shown. For $n \geq 2$, take a quotient $A' = A/I$ of A with $\mathrm{length}I = 1$ and $I^2 = \{0\}$. The kernel of $\rho \twoheadrightarrow \rho' = \rho \otimes_A A'$ is isomorphic to $\bar{\rho}$. Since D is an exact functor,

$$(\dagger) \qquad 0 \longrightarrow D(\bar{\rho}) \longrightarrow D(\rho) \longrightarrow D(\rho') \longrightarrow 0$$

is exact. Moreover, $(\dagger)$ is strictly compatible with the filtrations. This gives us an equality of the length

$$\mathrm{length}F^j(D(\rho)) = \mathrm{length}F^j(D(\rho')) + \mathrm{length}F^j(D(\bar{\rho}))$$

for $j = 0, 1$. By the assumption on the induction, $F^j(D(\rho'))$ is free of rank $2 - j$ over $o_F \otimes_{\mathbb{Z}_p} A'$ for $j = 0, 1$, so $F^j(D(\rho))$ is a quotient of $(o_F \otimes_{\mathbb{Z}_p} A)^{\oplus(2-j)}$ by Nakayama's Lemma. Since $\mathrm{length}F^j(D(\rho)) = \mathrm{length}(o_F \otimes_{\mathbb{Z}_p} A)^{\oplus(2-j)}$, we have the freeness of $F^j(D(\rho))$.

□

Definition 3.22. *Let A be an object of $\mathscr{C}_{o_\lambda}^{\mathrm{artin}}$ of characteristic p.*

(1) *$\mathscr{E}_A$ is the full subcategory of $M_{[0,1],A}^{FL}$ consisting of objects $D = (F^0(D), F^1(D), \varphi)$ which satisfy the following conditions:*
   - *$F^0(D)$ is free of rank 2 over $o_F \otimes_{\mathbb{Z}_p} A$.*
   - *$F^1(D)$ is free of rank 1 over $o_F \otimes_{\mathbb{Z}_p} A$.*

(2) *For $L_A = (o_F \otimes_{\mathbb{Z}_p} A)^{\oplus 2}$, we define a filtration by $F^0(L_A) = L_A$, $F^1(L_A) =$ the $A \otimes_{\mathbb{Z}_p} o_F$-submodule of $(o_F \otimes_{\mathbb{Z}_p} A)^{\oplus 2}$ generated by $(1, 0)$.*

   For an object $D = (F^0(D), F^1(D), \varphi)$ of $\mathscr{E}_A$, a frame f of D is an isomorphism $f : F^0(D) \simeq L_A$ as an $o_F \otimes_{\mathbb{Z}_p} A$-module which preserves the fltrations. (D, f) is called a framed object of $\mathscr{E}_A$. The set of all frames for D is denoted by $\mathrm{Frame}(D)$.

(3) *A framed isomorphism $(D_1, f_1) \to (D_2, f_2)$ is an isomorphism $\alpha : D_1 \to D_2$ in $M_{[0,1],A}^{FL}$ such that $f_2 \circ \alpha = f_1$.*

Remark 3.23. *By Definition 3.22 (3), a framed automorphism of a framed object of $\mathscr{E}_A$ reduces to the identity, and hence any framed isomorphisms between two framed objects are unique.*

For a flat deformation ρ of $\bar{\rho}$ with the trivial twist type, by Lemma 3.21, a frame always exists for $D(\rho)$.

Let G be the Weil restriction $\mathrm{Res}_{o_F/\mathbb{Z}_p} \mathrm{GL}_{2,o_F}$ of GL_{2,o_F}, B the Weil restriction of the standard Borel subgroup of GL_{2,o_F} consisting of the upper triangular matrices. These groups are smooth over $\mathbf{Z}_p$ since F is absolutely unramified.

For $D \in \mathrm{ob}\mathscr{E}_A$, $\mathrm{Frame}(D)$ forms a trivial $B(A)$-torsor. By Definition 3.22 (3), two frames f_1 and f_2 of D are isomorphic if and only if there is an automorphism $\alpha : D \simeq D$ in $M_{[0,1],A}^{FL}$ such that $f_2 = f_1 \circ \alpha$.

Lemma 3.24. *Let $\rho : G_F \to \mathrm{GL}_2(A)$ be a flat deformation of $\bar{\rho} : G_F \to \mathrm{GL}_2(k)$ with the trivial twist type.*

(1) *Assume that $pA = 0$. Let $X(A)$ be the set of framed isomorphism classes of framed objects of $\mathscr{E}_A$. If $X(A)$ is non-empty, it has a structure of a trivial torsor under $G(A)$.*
(2) *We fix a frame of $D(\bar\rho)$. For $A = k[\epsilon]$, the set $X_{\bar\rho}(A)$ of the framed isomorphism classes of framed objects D of $\mathscr{E}_A$ with a framed isomorphism $D \otimes_A k \simeq D(\bar\rho)$ has a structure of a* $\mathrm{Lie}G_k$*-torsor.*
(3) *We fix a frame of $D(\bar\rho)$. For $A = k[\epsilon]$, the set of the isomorphism classes of the frames of $D(\rho)$ lifting the frame of $D(\bar\rho)$ is a torsor under* $\mathrm{Lie}B_k/K$*, where K is a subgroup of* $\mathrm{Lie}B_k$ *which is canonically isomorphic to* $H^0(F, \mathrm{ad}\,\bar\rho)$.

Proof of Lemma 3.24. For (1), by the definition of $M^{FL}_{[0,1],A}$, $X(A)$ is identified with

$$X(A) = \{h : (o_F \otimes_{\mathbb{Z}_p} A)^2 \to (o_F \otimes_{\mathbb{Z}_p} A)^2,\ h \text{ is a } \sigma\text{-linear } A\text{-isomorphism.}\}$$

(1) is clear from this description with the $G(A)$-action given by $h \mapsto h \circ g^{-1}$ for $g \in G(A)$.

For (2), note that $X_{\bar\rho} = X_{\bar\rho}(A)$ is non-empty since it contains the constant deformation $\bar\rho \otimes_k A$. As in the case of (1), $X_{\bar\rho}$ has a structure of a trivial torsor under $\hat G(A)$, where $\hat G(A)$ is the subgroup of $G(A)$ consisting of the elements whose mod m_A-reduction is 1. $\hat G(A)$ is canonically isomorphic to $\mathrm{Lie}G_k$.

For (3), by Lemma 3.21, $D_\rho = D(\rho)$ admits a frame f_0. By changing it by an element of $B(A)$, we may assume that f_0 lifts the frame $\bar f_0$ of $D(\bar\rho)$. Then the set $\mathrm{Frame}(D_\rho)_{\bar f_0}$ of all frames of D_ρ lifting $\bar f_0$ is a torsor under $\hat B(A)$, where $\hat B(A) = \hat G(A) \cap B(A)$. $\hat B(A)$ is canonically isomorphic to $\mathrm{Lie}B_k$. To determine the isomorphism classes of the frames of D_ρ, it suffices to determine the automorphism group K of D_ρ whose mod m_A-reduction is the identity of $D(\bar\rho)$, since the isomorphism classes of the frames in $\mathrm{Frame}(D_\rho)_{\bar f_0}$ is identified with $\mathrm{Frame}(D_\rho)_{\bar f_0}/K \simeq \mathrm{Lie}B_k/K$. By the categorical equivalence, K is isomorphic to the group K' of the automorphisms of ρ which induce the identity on the mod m_A-reduction $\bar\rho$. It is well-known that K' is canonically isomorphic to $H^0(F, \mathrm{ad}\,\bar\rho)$. □

Proof of Theorem 3.20. We may assume that the twist character κ is trivial. $k = k_{o_\lambda}$, $A = k[\epsilon]$. We fix a frame of $D(\bar\rho)$. There is a canonical map

$$\pi : X_{\bar\rho} \longrightarrow F^{\mathbf{fl}}_{\bar\rho,\kappa}(k[\epsilon])$$

which associates the Galois module $V(D)_F$ to a framed object $(D, f) \in X_{\bar\rho}$. By Lemma 3.24 (3), π has a structure of $\mathrm{Lie}B_k/K$-torsor. By Lemma 3.24 (2), $X_{\bar\rho}$ is a $\mathrm{Lie}G_k$-torsor. Thus we have

$$\dim_k F^{\mathbf{fl}}_{\bar\rho,\kappa}(k[\epsilon]) = \dim_k \mathrm{Lie}G_k - \dim_k \mathrm{Lie}B_k + \dim_k K = [F : \mathbb{Q}_p] + \dim_k H^0(F, \mathrm{ad}\,\bar\rho).$$

For 3.20, (2), we just note that unramified twists $\rho \mapsto \rho \otimes \mu$ give a one dimensional family of deformations, so one should diminish 1 in the case of $F^{\mathbf{fl}}_{\bar\rho,\kappa,\chi}(k[\epsilon])$ because the determinant is fixed. □

3.7. ***K*-types.** The discussion in this paragraph localizes a global argument in [51], proposition 2.15. 0_E-case is analyzed as in [13], [9], though we make a modification using a result of [19] when the relative conductor is even.

First we recall a result of Géradin [19], which gives the local Langlands correspondence explicitly in some supercuspidal cases (cf. [13], §2, and [9], §4 for other applications to Hecke algebras).

Let $\tilde F$ be the unramified quadratic extension of F, σ the non-trivial element in $\mathrm{Gal}(\tilde F/F)$. Take a character $\psi : G_{\tilde F} \to E_\lambda^\times$ which does not extend to G_F. $\rho = \mathrm{Ind}_{G_{\tilde F}}^{G_F}\psi$. The conductor

c of ψ/ψ^σ depends only on ρ, and it is called the relative conductor of ρ. Here ψ^σ is the σ-twist of ψ, and $c \geq 1$ holds.

For an integer m, let $(m)_2 \in \{0,1\}$ be the integer which satisfies $m \equiv (m)_2 \mod 2$. $(m)_2$ is called the parity of m. For the relative conductor c, define a quaternion algebra D_c central over F by

$$D_c = \tilde{F} + \tilde{F}\Pi, \quad \Pi^2 = (p_F)^{(c)_2},\ \Pi x = \sigma(x)\Pi \quad (\forall x \in \tilde{F}),$$

and $G_c = D_c^\times$ is the multiplicative group of D_c.

D_c depends only on the parity of c, with the invariant $\frac{c}{2} \mod \mathbb{Z}$, and $o_{D_c} = o_{\tilde{F}} + o_{\tilde{F}}\Pi$ is a maximal order of D_c. We define an open subgroup K_c of $G_c(F)$ and a character μ_ψ of K_c by

$$K_c = \tilde{F}^\times(1 + m_{\tilde{F}}^d\Pi),$$
$$\mu_\psi|_{\tilde{F}^\times} = \psi \cdot \chi_{\tilde{F}},\ \mu_\psi|_{1+m_{\tilde{F}}^d\Pi} = \chi \circ \mathrm{N}_{D/F}.$$

Here $d = \frac{c-(c)_2}{2}$, $\chi_{\tilde{F}}$ is the unramified character corresponding to $\tilde{F}/F$, and $\mathrm{N}_{D/F}$ is the reduced norm (when $d = 0$, we regard $m_{\tilde{F}}^0$ as $o_{\tilde{F}}$). χ is a character of G_F such that $\psi \cdot \chi|_{G_{\tilde{F}}}{}^{-1}$ and ψ/ψ^σ has the same conductor (see [19], 3.2 for the existence of χ). ψ and χ are regarded as characters of $\tilde{F}^\times$ and $F^\times$ respectively by the local class field theory. μ_ψ does not depend on a choice of χ.

The induced representation $\mathrm{Ind}_{K_c}^{G_c(F)}\mu_\psi$ is irreducible and supercuspidal ([19], 3.4 and 5.1). Moreover, the representation π which corresponds to ρ by the local Langlands correspondence (resp. the local Langlands correspondence composed with the Jacquet-Langlands correspondence) is $\mathrm{Ind}_{K_c}^{G_c(F)}\mu_\psi$ when c is even (resp. odd) by [19], 3.8 and 5.4.

Assume that $\ell \neq p$, and we fix an ℓ-adic field E_λ. For $\bar{\rho} : G_F \to \mathrm{GL}_2(k_\lambda)$, and $* = \mathbf{f}, \mathbf{u}$, we attach an inner twist $G_{\bar{\rho}}$ of $\mathrm{GL}_{2,F}$, compact open subgroups $K_*(\bar{\rho})$ in $G_{\bar{\rho}}(F)$, and characters $\nu_*(\bar{\rho})$ of $K_*(\bar{\rho})$ having values in $o_{E_\lambda}^\times$.

In the case of 0_E, $\bar{\rho}$ is expressed as $\mathrm{Ind}_{G_{\tilde{F}}}^{G_F}\bar{\psi}$. By enlarging k_λ if neccesary, we may assume that $\bar{\psi}$ is defined over k_λ. Let $\psi = \bar{\psi}_{\mathrm{lift}} : G_{\tilde{F}} \to o_{E_\lambda}^\times$ be the Teichmüller lift of $\bar{\psi}$. We also view ψ as a character of $\tilde{F}^\times$ by the local class field theory.

Let c be the relative conductor of $\bar{\rho}$. Then $G_{\bar{\rho}} = G_c$,

$$K_{\mathbf{f}}(\bar{\rho}) = o_{\tilde{F}}^\times \cdot (1 + m_{\tilde{F}}^d\Pi),$$
$$K_{\mathbf{u}}(\bar{\rho}) = Z_{\tilde{F}}^\ell \cdot (1 + m_{\tilde{F}}^d\Pi).$$

Here $d = \frac{c-(c)_2}{2}$, and $Z_{\tilde{F}}^\ell$ is the minimal subgroup of $o_{\tilde{F}}^\times$ containing $1 + m_{\tilde{F}}$ of ℓ-power index. $\nu_*(\bar{\rho}) : K_*(\bar{\rho}) \to o_{E_\lambda}^\times$ is the restriction of μ_ψ to $K_*(\bar{\rho})$ for $* = \mathbf{f}, \mathbf{u}$.

Next we treat the cases other than 0_E. When $\bar{\rho}$ is absolutely reducible, by enlarging k_λ if necessary, we may assume that $\bar{\rho}$ is reducible over k_λ, and $\bar{\kappa}$ be the twist type of $\bar{\rho}$. In the case of 0_{NE}, we define the twist type $\bar{\kappa}$ of $\bar{\rho}$ as the trivial character.

We define the integers $c(\bar{\rho})$ and $d(\bar{\rho})$ by

$$c(\bar{\rho}) = \mathrm{Art}(\bar{\rho}|_{I_F} \otimes \bar{\kappa}^{-1}),$$
$$d(\bar{\rho}) = \dim_{k_\lambda}(\bar{\rho}|_{I_F} \otimes \bar{\kappa}^{-1})^{I_F}.$$

Here Art denotes the Artin conductor. $G_{\bar{\rho}}$ is $\mathrm{GL}_{2,F}$, and $K_{\mathbf{f}}(\bar{\rho})$ and $K_{\mathbf{u}}(\bar{\rho})$ are defined by

$$K_{\mathbf{f}}(\bar{\rho}) = \mu_{\ell^\infty}(F) \cdot K_1(m_F^{c(\bar{\rho})}),$$

$$K_{\mathbf{u}}(\bar\rho) = K_1(m_F^{c(\bar\rho)+d(\bar\rho)}).$$

For $* = \mathbf{f}, \mathbf{u}$, the K-character $\nu_*(\bar\rho) : K_*(\bar\rho) \to o_{E_\lambda}^\times$ of $\bar\rho$ is the composition of

$$K_*(\bar\rho) \longrightarrow \mathrm{GL}_2(o_F) \xrightarrow{\det} o_F^\times \xrightarrow{\kappa} o_{E_\lambda}^\times.$$

Here κ is the Teichmüller lift of the twist type $\bar\kappa$ of $\bar\rho$, which is regarded as a character of $o_F^\times$ by the local class field theory.

Definition 3.25. *Assume that $\ell \neq p$, and let $\bar\rho : G_F \to \mathrm{GL}_2(k_\lambda)$ be a G_F-representation. The data $(G_{\bar\rho}, (K_*(\bar\rho), \nu_*(\bar\rho))_{*\in\{\mathbf{f},\mathbf{u}\}})$ defined for $\bar\rho$ is called the type of $\bar\rho$. $\nu_*(\bar\rho)$ is called the K-character of $\bar\rho$.*

Take a deformation ρ of $\bar\rho$ over o_{E_λ}. Let $\rho_{E_\lambda}^{WD} : W_F' \to \mathrm{GL}_2(E_\lambda)$ be the representation of the Weil-Deligne group W_F' of F associated to ρ_{E_λ}, and π the irreducible admissible representation of $\mathrm{GL}_2(F)$ defined over $\bar E_\lambda$ associated to the F-semi-simplification of $\rho_{E_\lambda}^{WD}$ by the local Langlands correspondence. If $G_{\bar\rho}$ is isomorphic to $\mathrm{GL}_{2,F}$ (resp. the multiplicative group of a division quaternion algebra), we denote π (resp. the Jacquet-Langlands correspondent of π) by π_ρ.

Definition 3.26. *Assume that $\ell \neq p$, and let $\bar\rho : G_F \to \mathrm{GL}_2(k_\lambda)$ be a G_F-representation of type $(G_{\bar\rho}, (K_*(\bar\rho), \nu_*(\bar\rho))_{*\in\{\mathbf{f},\mathbf{u}\}})$, and ρ a deformation of $\bar\rho$ over o_{E_λ}.*

1. *An irreducible admissible representation π of $G_{\bar\rho}(F)$ defined over $\bar E_\lambda$ is associated to ρ if it is isomorphic to π_ρ.*
2. *For an admissible irreducible representation $\pi : G_{\bar\rho}(F) \to \mathrm{Aut}_{\bar E_\lambda} V$ associated to ρ, and for $* = \mathbf{f}, \mathbf{u}$,*
$$I_*(\bar\rho, \pi) = \mathrm{Hom}_{K_*(\bar\rho)}(\nu_*(\bar\rho),\ V).$$

Proposition 3.27. *Assume that $\ell \neq p$, and let $\bar\rho : G_F \to \mathrm{GL}_2(k_\lambda)$ be a G_F-representation of type $(G_{\bar\rho}, (K_*(\bar\rho), \nu_*(\bar\rho))_{*\in\{\mathbf{f},\mathbf{u}\}})$, ρ a deformation of $\bar\rho$ over o_{E_λ}, and π an admissible irreducible representation associated to ρ.*

Then $I_{\mathbf{f}}(\bar\rho, \pi)$ is at most one dimensional. It is non-zero if and only if ρ is a finite deformation of $\bar\rho$.

Proof of Proposition 3.27. Note that $K_{\mathbf{f}}(\bar\rho)$ contains $\mu_{\ell^\infty}(F)$, so $\det\rho_\pi$ is a finite deformation of $\det\bar\rho$.

First we treat the case of 0_E. Assume that $I_{\mathbf{f}}(\bar\rho, \pi) \neq \{0\}$. $\bar\rho = \mathrm{Ind}_{G_{\tilde F}}^{G_F} \bar\psi$, ψ the Teichmüller lift of $\bar\psi$. The relative conductor of ψ is c. Let V be the representation space of π. $I_{\mathbf{f}}(\bar\rho, \pi)$ is identified with the subspace of $V^{\ker \nu_{\mathbf{f}}(\bar\rho)}$ where $K_{\mathbf{f}}(\bar\rho)$ acts as $\nu_{\mathbf{f}}(\bar\rho)$. By twisting by an unramified character, we may assume that the central element p_F acts on V by $\mu_\psi(p_F)$. So we may assume that the action of $p_F^{\mathbb{Z}} \cdot K_{\mathbf{f}}(\bar\rho) = K_c$ on $I_{\mathbf{f}}(\bar\rho, \pi) \subset V^{\ker \nu_{\mathbf{f}}(\bar\rho)}$ is μ_ψ.

Since $K_{\mathbf{f}}(\bar\rho)$ is a compact open subgroup and π is admissible, the $K_{\mathbf{f}}(\bar\rho)$-action on V is semi-simple, and hence $I_{\mathbf{f}}(\bar\rho, \pi)$ appears as a quotient representation of $\pi|_{K_{\mathbf{f}}(\bar\rho)}$. By the Frobenius reciprocity,

$$\dim_{\bar E_\lambda} \mathrm{Hom}_{K_c}(\pi|_{K_c}, \mu_\psi) = \dim_{\bar E_\lambda} I_{\mathbf{f}}(\bar\rho, \pi)^\vee \cdot \dim_{\bar E_\lambda} \mathrm{Hom}_{G_{\bar\rho}(F)}(\pi, \mathrm{Ind}_{K_c}^{G_{\bar\rho}(F)} \mu_\psi).$$

Since π and $\mathrm{Ind}_{K_c}^{G_{\bar\rho}(F)} \mu_\psi$ are both irreducible, they are isomorphic, and $\dim_{\bar E_\lambda} I_{\mathbf{f}}(\bar\rho, \pi) = 1$ by Schur's lemma. $\rho_{E\lambda}$ is isomorphic to $\mathrm{Ind}_{G_{\tilde F}}^{G_F} \psi$, and hence is a finite deformation of $\bar\rho$. The argument also shows that $\dim_{\bar E_\lambda} I_{\mathbf{f}}(\bar\rho, \pi) = 1$ in the case of finite deformations.

In the other cases, let ν be a character of $F^\times$ such that $\nu|_{I_F}$ is the Teichmüller lift $\bar\kappa_{\mathrm{lift}}$ of twist character κ. By replacing π by $\pi \otimes \nu^{-1}$, we may assume that $\nu_{\mathbf{f}}(\bar\rho)$ is trivial. Let

$c(\pi)$ be the conductor of π. Assume that $I_{\mathbf{f}}(\bar\rho,\pi)\neq\{0\}$. Since $K_{\mathbf{f}}(\bar\rho)$ contains $K_1(m_F^{c(\bar\rho)})$,

$$c(\bar\rho)\geq c(\pi)$$

since π has a non-zero $K_1(m^{c(\bar\rho)})$-fixed vector. $c(\pi)$ is equal to $\mathrm{Art}\rho_{E_\lambda}$ since π is associated to ρ (the local Langlands correspondence preserves the Galois and automorphic conductors), and hence

$$\mathrm{Art}\bar\rho\geq\mathrm{Art}\rho_{E_\lambda}.$$

On the other hand, for the Artin conductors

$$\mathrm{Art}\bar\rho-\mathrm{Art}\rho_{E_\lambda}=\dim_{E_\lambda}(\rho_{E_\lambda}|_{I_F})^{I_F}-\dim_{k_\lambda}(\bar\rho|_{I_F})^{I_F}\leq 0$$

holds. Thus we have $\mathrm{Art}\rho_{E_\lambda}=\mathrm{Art}\bar\rho$, and hence the equality $\dim_{k_\lambda}(\rho_{E_\lambda}|_{I_F})^{I_F}=\dim_{E_\lambda}(\bar\rho|_{I_F})^{I_F}$. ρ^{I_F} is o_{E_λ}-free and is an o_{E_λ}-direct summand, and hence the last equality implies that ρ is a finite deformation of $\bar\rho$.

If ρ is a finite deformation, by an argument as above $c(\pi)=\mathrm{Art}\bar\rho$ holds. $\pi^{K_{\mathbf{f}}(\bar\rho)}=\pi^{K_1(m_F^{c(\pi)})}$ is the space of new vectors, and hence $I_{\mathbf{f}}(\bar\rho,\pi)$ is one dimensional. □

Let ρ be a deformation of $\bar\rho$ over o_{E_λ}, and π the admissible representation associated to ρ. Assume that $\bar\rho$ is not of type 0_E. In the unrestricted case, we need to choose a one dimensional subspace from $I_{\mathbf{u}}(\bar\rho,\pi)$. We use the $U(p_F)$-operator for this purpose ([51]) .

In general, for a reductive group G and a compact open subgroup K of $G(F)$, let $H(G(F),K)_{\bar E_\lambda}$ be the convolution algebra formed by the $\bar E_\lambda$-valued compactly supported K-biinvariant functions on $G(F)$. For any irreducible admissible representation π of $G(F)$ defined over $\bar E_\lambda$, $H(G(F),K)_{\bar E_\lambda}$ acts on π^K.

Definition 3.28. *Let π be an irreducible admissible representation of $\mathrm{GL}_2(F)$ defined over $\bar E_\lambda$. For a uniformizer p_F of F,*

$$U(p_F),\ U(p_F,p_F):\pi^K\longrightarrow\pi^K$$

are defined by the characteristic functions of the double cosets $K\begin{pmatrix}1&0\\0&p_F\end{pmatrix}K$ and $K\begin{pmatrix}p_F&0\\0&p_F\end{pmatrix}K$, repectively.

Note that $U(p_F)$ and $U(p_F,p_F)$-operators thus defined may depend on a choice of a uniformizer.

In our situation, we choose $K=\ker\nu_{\mathbf{u}}(\bar\rho)$. $U(p_F)$ acts on $I_{\mathbf{u}}(\bar\rho,\pi)$ by identifying $I_{\mathbf{u}}(\bar\rho,\pi)$ as a subspace of π^K.

Proposition 3.29. *Assume that $\ell\neq p$, and let $\bar\rho:G_F\to\mathrm{GL}_2(k_\lambda)$ be a G_F-representation of type $(G_{\bar\rho},(K_*(\bar\rho),\nu_*(\bar\rho))_{*\in\{\mathbf{f},\mathbf{u}\}})$, ρ a deformation of $\bar\rho$ over o_{E_λ}, and π an admissible irreducible representation associated to ρ. $\kappa=\bar\kappa_{\mathrm{lift}}$ is the Teichmüller lift of the twist character of $\bar\rho$.*

1. *If $\bar\rho$ is of type 0_E, $I_{\mathbf{u}}(\bar\rho,\pi)$ is one dimensional.*
2. *If $\bar\rho$ is not of type 0_E, then $I_{\mathbf{u}}(\bar\rho,\pi)$ is isomorphic to*

$$\mathscr{L}_{\bar\rho,p_F}=\bar E_\lambda[U]/(U\cdot L(U,\ \pi\otimes(\nu_{\bar\rho,p_F}\circ\det)^{-1}))$$

as an $\bar E_\lambda[U]$-module. Here U acts as $U(p_F)$ on $I_{\mathbf{u}}(\bar\rho,\pi)$, $\nu_{\bar\rho,p_F}$ is the character of $F^\times$ such that $\nu_{\bar\rho,p_F}(p_F)=1$ and $\nu_{\bar\rho,p_F}|_{o_F^\times}=\kappa$, and $L(T,\ \pi\otimes(\nu_{\bar\rho,p_F}\circ\det)^{-1})$ is the standard L-function of $\pi\otimes(\nu_{\bar\rho,p_F}\circ\det)^{-1}$. The localization of $\mathscr{L}_{\bar\rho,p_F}$ at $U=0$ is one dimensional.

Proof of Proposition 3.29. For (1), first show that $I_{\mathbf{u}}(\bar\rho,\pi)\neq\{0\}$. Since ρ is a deformation of $\bar\rho$, $\rho_{\bar E_\lambda}$ is written as $\mathrm{Ind}_{G_{\tilde F}}^{G_F}\phi$, where ϕ is a character of $G_{\tilde F}$. We may assume that ϕ is a lift of $\bar\psi$. The relative conductor c of ϕ is the same as that of $\bar\psi$. By the result of Géradin, π is isomorphic to $\mathrm{Ind}_{K_c}^{G_{\bar\rho}(F)}\mu_\phi$, and $\pi|_{K_c}$ contains a non-zero subspace W where K_c acts by μ_ϕ. It is clear that $W\subset I_{\mathbf{u}}(\bar\rho,\pi)$.

$I_{\mathbf{u}}(\bar\rho,\pi)$ is identified with the subspace of $V^{\ker\nu_{\mathbf{u}}(\bar\rho)}$ where $K_{\mathbf{u}}(\bar\rho)$ acts as $\nu_{\mathbf{u}}(\bar\rho)$. We consider the action of $\mu_{\ell^\infty}(\tilde F)$ on $I_{\mathbf{u}}(\bar\rho)$, and let α be a character of $\mu_{\ell^\infty}(\tilde F)$ which appears as a subrepresentation of $I_{\mathbf{u}}(\bar\rho,\pi)$. We identify α with a character of $k_{\tilde F}^\times$ of ℓ-power order. Let π_ψ be the $G(F)$-representation corresponding to $\mathrm{Ind}_{G_{\tilde F}}^{G_F}\psi$, β be the character of $o_D^\times$ defined by $o_D^\times\to o_D^\times/(1+\Pi o_D)=o_{\tilde F}^\times/(1+m_{\tilde F})=k_{\tilde F}^\times\xrightarrow{\alpha}o_{E_\lambda}^\times$. By twisting by an unramified character, we may assume that the central character of π takes the same value as the central character of π_ψ at p_F. The action of $\tilde F^\times$ is thus determined, and it follows that there is a subspace of $I_{\mathbf{u}}(\bar\rho,\pi)$ where K_c acts by $\mu_\psi\cdot\beta|_{K_c}$.

By the definition, $\mu_{\psi\cdot\alpha}=\mu_\psi\cdot\beta|_{K_c}$, and hence π corresponds to $\mathrm{Ind}_{G_{\tilde F}}^{G_F}(\psi\cdot\alpha)$ by the local Langlands correspondence. This shows the uniqueness of α, and hence K_c acts on $I_{\mathbf{u}}(\bar\rho,\pi)$ by $\mu_{\psi\cdot\alpha}$. The calculation of the dimension is the same as in the case of $I_{\mathbf{f}}(\bar\rho,\pi)$ by using the Frobenius reciprocity.

For (2), we may assume that $\bar\kappa$ is trivial by twisting by a character of order prime to ℓ.

$$\mathrm{Art}\rho_{E_\lambda}+\dim_{E_\lambda}\rho_{E_\lambda}^{I_F}=2+\mathrm{sw}\,\rho_{E_\lambda}=2+\mathrm{sw}\,\bar\rho=\mathrm{Art}\bar\rho+\dim_{k_\lambda}\bar\rho^{I_F}=c(\bar\rho)+d(\bar\rho)$$

by the formula for Artin conductors. Here sw means the swan conductor, which remains unchanged by a mod λ-reduction. $\dim_{E_\lambda}\rho_{E_\lambda}^{I_F}=\deg L(T,\pi)$ and $\mathrm{Art}\rho_{E_\lambda}=\mathrm{cond}\pi$, since the local Langlands correspondence preserves the L and ϵ-factors. Thus 3.29, (2) is reduced to the following lemma.

Lemma 3.30. *For an admissible irreducible representation π of $\mathrm{GL}_2(F)$ defined over $\bar E_\lambda$, let c be the conductor of π, d the degree of $L(T,\ \pi)$, where $L(T,\ \pi)$ is the standard L-function of π. We regard $\pi^{K_1(m^{c+d})}$ as an $\bar E_\lambda[U]$-module, where the U-action is given by the $U(p_F)$-operator. Then it is isomorphic to $\bar E_\lambda[U]/(U\cdot L(U,\ \pi))$.*

Proof of Lemma 3.30. When π is supercuspidal, this is well-known. Let v be a non-zero vector in $\pi^{K_1(m^c)}$ (new vector). Then $\pi^{K_1(m^{c+d})}$ has a basis by $\begin{pmatrix}p_F&0\\0&1\end{pmatrix}^i v,\ 0\le i\le d$. By writing down the $U(p_F)$-action explicitly, the lemma follows. We omit the details (for $F=\mathbb{Q}$ and in the global setting, this is found in [51]). $\square$

$\square$

3.8. **Global deformations.** In this subsection, F is a global field. For a continuous representation ρ of the global Galois group $G_F=\mathrm{Gal}(\bar F/F)$ and a finite place v of F, we denote the restriction to the decomposition group $\rho|_{G_{F_v}}$ by $\rho|_{F_v}$. For a finite set of finite places Σ, $G_\Sigma=\pi_1^{\text{ét}}(\mathrm{Spec}\,o_F\setminus\Sigma)$ be the Galois group of the maximal Galois extension of F which is unramified outside Σ.

Let k be a finite field, $\bar\rho:G_F\to\mathrm{GL}_2(k)$ be an absolutely irreducible representation of G_F. Assume the following conditions:

- **DC1** For $v|\ell$, $\bar\rho|_{F_v}$ is flat or nearly ordinary. If $\bar\rho|_{F_v}$ is flat (resp. nearly ordinary) at v, a flat twist type (resp. nearly ordinary type) $\bar\kappa_v$ is specified, and is defined over k.
- **DC2** For $v\nmid\ell$ and $\bar\rho|_{F_v}$ is absolutely reducible, it is reducible over k, and a twist type $\bar\kappa_v$ (Definition 3.1) is specified, and is defined over k.

DC3 For $v \nmid \ell$ and $\bar{\rho}|_{F_v}$ is of type 0_E, an inertia character $\bar{\psi}$ (Definition 3.1) is specified, and is defined over k.

Definition 3.31. *Let $\bar{\rho} : G_F \to \mathrm{GL}_2(k)$ be an irreducible mod ℓ-representation which satisfies* **DC1-3**.

(1) *A deformation function d of $\bar{\rho}$ is a map $|F|_f \to \{\mathbf{f}, \mathbf{u}, \mathbf{n.o.f.}, \mathbf{n.o.}, \mathbf{fl}\}$ which satisfies the following properties:*
- *If $v \nmid \ell$, then $d(v) \in \{\mathbf{f}, \mathbf{u}\}$.*
- *If $v|\ell$ and $\bar{\rho}|_{F_v}$ is nearly ordinary, $d(v) \in \{\mathbf{n.o.f.}, \mathbf{n.o.}\}$.*
- *If $v|\ell$ and $\bar{\rho}|_{F_v}$ is flat, $d(v) = \mathbf{fl}$.*
- *For almost all places v of F, $d(v) = \mathbf{f}$.*

(2) *A deformation type $\mathscr{D}$ of $\bar{\rho}$ is a quartet $(d,\ o_\lambda, \{\kappa_v\}_{v \in \mathbf{NO}(\bar{\rho})}, \{\kappa_v\}_{v \in \mathbf{FL}(\bar{\rho})})$ such that*
- *d is a deformation function of $\bar{\rho}$. d is called the deformation function of $\mathscr{D}$, and denoted by $\mathrm{def}_{\mathscr{D}}$.*
- *o_λ is a complete noetherian local ring with the maximal ideal m_{o_λ} and the residue field k_{o_λ}. k_{o_λ} is isomorphic to k. o_λ is called the coefficient ring of $\mathscr{D}$, and denoted by $o_{\mathscr{D}}$.*
- *$\mathbf{NO}(\bar{\rho})$ (resp. $\mathbf{FL}(\bar{\rho})$) is the set of finite places where $\rho|_{F_v}$ is nearly ordinary with the nearly ordinary type $\bar{\kappa}_v$ (resp. flat with the twist character $\bar{\kappa}_v$). $\kappa_v : G_{F_v} \to o_\lambda^\times$ for $v \in \mathbf{NO}(\bar{\rho})$ (resp. $v \in \mathbf{FL}(\bar{\rho})$) is a continuous character such that $\kappa_v \mod m_{o_\lambda} = \bar{\kappa}_v$. For $v \in \mathbf{NO}(\bar{\rho})$ (resp. $v \in \mathbf{FL}(\bar{\rho})$), κ_v is called the nearly ordinary type of $\mathscr{D}$ (resp. flat twist type of $\mathscr{D}$), and is denoted by $\kappa_{\mathscr{D},v}$.*

(3) *The ramification set $\Sigma_{\mathscr{D}}$ of a deformation type $\mathscr{D}$ of $\bar{\rho}$ is*

$$\Sigma_{\mathscr{D}} = \{v|\ell\} \cup \{v : \bar{\rho} \text{ is ramified at } v\} \cup \{v : \bar{\rho} \text{ is unramified at } v,\ \mathrm{def}_{\mathscr{D}}(v) = \mathbf{u}\}.$$

Definition 3.32. (1) *For a deformation type $\mathscr{D} = (d,\ o_\lambda, \{\kappa_v\}_{v \in \mathbf{NO}(\bar{\rho})}, \{\kappa_v\}_{v \in \mathbf{FL}(\bar{\rho})})$ of $\bar{\rho}$ and a local homomorphism $o_\lambda \to o'_{\lambda'}$ of complete noetherian local rings, the scalar extension $\mathscr{D}_{o'_{\lambda'}}$ is defined by $(d,\ o'_{\lambda'}, \{\kappa'_v\}_{v \in \mathbf{NO}(\bar{\rho})}, \{\kappa'_v\}_{v \in \mathbf{FL}(\bar{\rho})})$. Here κ'_v is the composition of $G_{F_v} \xrightarrow{\kappa_v} o_\lambda^\times \to (o'_{\lambda'})^\times$ for $v \in \mathbf{NO}(\bar{\rho})$ (resp. $v \in \mathbf{FL}(\bar{\rho})$).*

(2) *We define a partial order $\le$ on the set $\{\mathbf{f}, \mathbf{u}, \mathbf{n.o.f.}, \mathbf{n.o.}, \mathbf{fl}\}$ by $\mathbf{f} \le \mathbf{u}$, $\mathbf{n.o.f} \le \mathbf{n.o.}$, $\mathbf{fl} \le \mathbf{fl}$. A deformation type $\mathscr{D}$ of $\bar{\rho}$ is minimal if $\mathrm{def}_{\mathscr{D}}(v)$ takes the minimal possible value at any $v \in |F|_f$ for the partial order.*

(3) *A morphism $\mathscr{D} \to \mathscr{D}'$ between deformation types of $\bar{\rho}$ is a local ring homomorphism $f : o_{\mathscr{D}'} \to o_{\mathscr{D}}$ and a condition on deformation functions which satisfy the following properties:*
- *$G_{F_v} \xrightarrow{\kappa_{\mathscr{D}',v}} o_{\mathscr{D}'}^\times \xrightarrow{f} o_{\mathscr{D}}^\times$ is $\kappa_{\mathscr{D},v}$ for $v \in \mathbf{NO}(\bar{\rho}) \cup \mathbf{FL}(\bar{\rho})$.*
- *$\mathrm{def}_{\mathscr{D}}(v) \le \mathrm{def}_{\mathscr{D}'}(v)$ for any $v \in |F|_f$.*

The category of deformation types of $\bar{\rho}$ is denoted by $\mathrm{Type}(\bar{\rho})$.

We now define the global deformation functor associated with a deformation type $\mathscr{D}$ of $\bar{\rho}$.

Definition 3.33. *Let $\bar{\rho} : G_F \to \mathrm{GL}_2(k)$ be an absolutely irreducible representation over a finite field of characteristic ℓ, $\mathscr{D}$ a deformation type of $\bar{\rho}$.*

For the coefficient ring $o_{\mathscr{D}}$ of $\mathscr{D}$ and $A \in \mathrm{ob}\mathscr{C}_{o_{\mathscr{D}}}^{\mathrm{noeth}}$, $\rho : G_{\Sigma_{\mathscr{D}}} \to \mathrm{GL}_2(A)$ is a deformation of $\bar{\rho}$ of type $\mathscr{D}$ if the following conditions are satisfied:
- *$\rho \mod m_A \simeq \bar{\rho}$.*
- *If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{f}$ (resp. $\mathbf{u}$) at $v \in |F|_f$, $\rho|_{F_v}$ is a finite (resp. unrestricted) deformation of $\bar{\rho}|_{F_v}$.*
- *If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{n.o.f}$. (resp. $\mathbf{n.o.}$) at $v \in |F|_f$, $\rho|_{F_v}$ is a nearly ordinary finite (resp. nearly ordinary) deformation of $\bar{\rho}|_{F_v}$ of nearly ordinary type $\kappa_{\mathscr{D},v}$.*

- *If* $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{fl}$ *at* $v \in |F|_f$, $\rho|_{F_v}$ *is a flat deformation of* $\bar{\rho}|_{F_v}$ *of twist type* $\kappa_{\mathscr{D},v}$.

By $F_{\mathscr{D}}(A)$ *we denote the set of isomorphism classes of deformations of* $\bar{\rho}$ *of type* $\mathscr{D}$ *over* A, *and the functor*

$$F_{\mathscr{D}} : \mathscr{C}^{\mathrm{noeth}}_{o_{\mathscr{D}}} \longrightarrow \mathbf{Sets}$$

is the deformation functor of $\bar{\rho}$ *of type* $\mathscr{D}$.

For a continuous character $\chi : G_{\Sigma_{\mathscr{D}}} \to o_{\mathscr{D}}^{\times}$, $F_{\mathscr{D},\chi}$ *is the subfunctor of* $F_{\mathscr{D}}$ *consisting of the deformations with the determinant* χ.

Theorem 3.34. *Let* $\bar{\rho} : G_F \to \mathrm{GL}_2(k)$ *be an absolutely irreducible representation over a finite field of characteristic* ℓ, $\mathscr{D}$ *a deformation type of* $\bar{\rho}$, $\chi : G_{\Sigma_{\mathscr{D}}} \to o_{\mathscr{D}}^{\times}$ *a continuous character. Then the deformation functor* $F_{\mathscr{D}}$ *(resp.* $F_{\mathscr{D},\chi}$*) of* $\bar{\rho}$ *is representable.*

The representability follows as in [32], by the use of the Grothendieck-Schlessinger criterion, using $H^0(G_\Sigma, \mathrm{ad}^0\,\bar{\rho}) = \{0\}$ to assure the universality. The proof is so standard, and the details are omitted.

By $R_{\mathscr{D}}$ (resp. $R_{\mathscr{D},\chi}$), we denote the universal deformation ring representing $F_{\mathscr{D}}$ (resp. $F_{\mathscr{D},\chi}$). $\rho^{\mathrm{univ}}_{\mathscr{D}} : G_{\Sigma_{\mathscr{D}}} \to \mathrm{GL}_2(R_{\mathscr{D}})$ (resp. $\rho^{\mathrm{univ}}_{\mathscr{D},\chi} : G_{\Sigma_{\mathscr{D}}} \to \mathrm{GL}_2(R_{\mathscr{D},\chi})$ is the universal representation.

As in [51], the universal deformation ring behaves nicely under the change of the coefficient ring by $o_\lambda \to o'_{\lambda'}$: $R_{\mathscr{D}_{o'_{\lambda'}}} \xrightarrow{\sim} R_{\mathscr{D}} \otimes_{o_\lambda} o'_{\lambda'}$.

By the discussions in §3.3 and §3.6, we have

Proposition 3.35. (1) *The tangent space* $F_{\mathscr{D}}(k_{o_{\mathscr{D}}}[\epsilon])$, *which is identified with the Zariski tangent space* $\mathrm{Hom}_{k_{o_{\mathscr{D}}}}(m_{R_{\mathscr{D}}}/(m^2_{R_{\mathscr{D}}}, m_{o_{\mathscr{D}}}), k_{o_{\mathscr{D}}})$ *of* $R_{\mathscr{D}}$ *over* $o_{\mathscr{D}}$, *is canonically isomorphic to*

$$H^1_{\mathscr{D}}(F,\ \mathrm{ad}\,\bar{\rho}) = \ker(H^1(F,\ \mathrm{ad}\,\bar{\rho}) \longrightarrow \bigoplus_{v \in |F|_f} H^1(F_v,\ \mathrm{ad}\,\bar{\rho}|_{F_v})/L_v).$$

Here L_v *is the local tangent space for* $\bar{\rho}|_{F_v}$ *defined by* $F^{\mathrm{def}_{\mathscr{D}}(v)}_{\bar{\rho}|_{F_v},\kappa_{\mathscr{D},v}}(k_{o_{\mathscr{D}}}[\epsilon])$ *if* $\mathrm{def}_{\mathscr{D}}(v) \in \{\mathbf{n.o.f}, \mathbf{n.o.}, \mathbf{fl}\}$, $L_v = F^{\mathrm{def}_{\mathscr{D}}(v)}_{\bar{\rho}|_{F_v s}}(k_{o_{\mathscr{D}}}[\epsilon])$ *in the other cases.*

(2) *For a continuous character* $\chi : G_F \to o_{\mathscr{D}}^{\times}$, *the statement corresponding to (1) holds for* $F_{\mathscr{D},\chi}$ *and* $R_{\mathscr{D},\chi}$ *using* $\mathrm{ad}^0\,\bar{\rho}$ *instead of* $\mathrm{ad}\,\bar{\rho}$.

3.9. **Selmer groups.** To apply the level raising formalism in §2, we need to use a variant of Selmer group associated to $\mathrm{ad}\,\rho$ as in [51]. Since it is an easy translation of §1 of [51], we briefly discuss it here for our later use.

For an absolutely irreducible representation $\bar{\rho}$ with a deformation type $\mathscr{D}$, let $R_{\mathscr{D}}$ be the universal deformation ring. We assume that the coefficient ring $o_{\mathscr{D}}$ is an ℓ-adic integer ring with the fraction field $E_{\mathscr{D}}$, and that there is an $o_{\mathscr{D}}$-homomorphism $f : R_{\mathscr{D}} \to o_{\mathscr{D}}$. By the universality of $R_{\mathscr{D}}$, f corresponds to a deformation $\rho : G_{\Sigma_{\mathscr{D}}} \to \mathrm{GL}_2(o_{\mathscr{D}})$ of $\bar{\rho}$ of type $\mathscr{D}$.

Let $L \simeq o_{\mathscr{D}}^{\oplus 2}$ be the representation space of ρ. We define M and M^0 by

$$M = \mathrm{ad}\,\rho = L^{\vee} \otimes_{o_{\mathscr{D}}} L,$$

$$M^0 = \mathrm{ad}^0\,\rho.$$

For each integer $n \geq 1$,

$$M_n = M \otimes_{o_{\mathscr{D}}} m_{\mathscr{D}}^{-n}/o_{\mathscr{D}}.$$

Note that the natural inclusion $M_n \hookrightarrow M \otimes_{o_{\mathscr{D}}} E_{\mathscr{D}}/o_{\mathscr{D}}$ induces an injection

$$H^1(F,\ M_n) \longrightarrow H^1(F,\ M \otimes_{o_{\mathscr{D}}} E_{\mathscr{D}}/o_{\mathscr{D}})$$

since $H^0(F,\ \mathrm{ad}^0\,\bar\rho)=\{0\}$.

For a finite place v and an integer n, we define a local subgroup $H^1_{\mathrm{def}_{\mathscr{D}}(v)}(F_v,\ M_n)\subset H^1(F_v,\ M_n)$ ([51], [9], p.142) which reduces to the local tangent space if $n=1$.

First assume that $\mathrm{def}_{\mathscr{D}}(v)=\mathbf{n.o.}$. Let $\kappa_{\mathscr{D},v}$ be the nearly ordinary type of $\bar\rho$. $\rho|_{F_v}$ has the form

$$0\longrightarrow\chi_{1,v}\longrightarrow\rho|_{F_v}\longrightarrow\chi_{2,v}\longrightarrow 0.$$

Here $\chi_{1,v}$ is the nearly ordinary character of $\rho|_{F_v}$.

As in §3.5, we define $W_1M_n^0\subset M_n^0$ by

$$W_1M_n^0=\rho|_{F_v}\otimes\chi_{2,v}^{-1}.$$

There is an exact sequence

$$0\longrightarrow W_2M_n^0\longrightarrow W_1M_n^0\longrightarrow\mathscr{O}_n\longrightarrow 0$$

for $W_1M_n^0$. Here $\mathscr{O}_n=o_{\mathscr{D}}/m_{\mathscr{D}}^n$, and $W_2M_n^0=\chi_{1,v}\cdot\chi_{2,v}^{-1}\mod m_{\mathscr{D}}^n$.

We define a subgroup $N_{\mathbf{n.o.},n}$ of $H^1(F_v^{\mathrm{unr}},M_n^0)$ as the image of $H^1(F_v^{\mathrm{unr}},W_2M_n^0)$ induced by $W_2M_n^0\to M_n^0$. Then

$$H^1_{\mathbf{n.o.}}(F_v,\ M_n^0)=\ker(H^1(F_v,\ M_n^0)\to H^1(F_v^{\mathrm{unr}},\ M_n^0)/N_{\mathbf{n.o.},n}),$$

$$H^1_{\mathbf{n.o.}}(F_v,\ M_n)=H^1_{\mathbf{n.o.}}(F_v,\ M_n^0)\oplus H^1(F_v^{\mathrm{unr}}/F_v,\mathscr{O}_n).$$

Let $\mathscr{O}_n[\epsilon]$ be the ring of dual numbers over $\mathscr{O}_n$. Using that $\bar\rho$ is G_{F_v}-distinguished, as in [51], proposition 1.1, we may regard $H^1_{\mathbf{n.o.}}(F_v,\ M_n^0)$ as the group of extensions of the form

$$0\longrightarrow\tilde\chi_{1,v}\longrightarrow\tilde\rho\longrightarrow\tilde\chi_{2,v}\longrightarrow 0$$

such that $\tilde\rho\mod\epsilon\simeq\rho|_{F_v}\mod m_{\mathscr{D}}^n$, and $\det\tilde\rho=\det\rho|_{F_v}$. Here $\tilde\chi_{i,v}:G_{F_v}\to\mathscr{O}_n[\epsilon]^\times$ is a character which lifts $\chi_{i,v}\mod m_{\mathscr{D}}^n$ for $i=1,2$, $\chi_{1,v}|_{I_{F_v}}$ is equal to the nearly ordinary type $\kappa_{\mathscr{D},v}$.

Next assume that $\mathrm{def}_{\mathscr{D}}(v)=\mathbf{n.o.f.}$. $\chi_1|_{I_{F_v}}=\chi_2(1)|_{I_{F_v}}$ holds by the definition, and hence $\rho|_{F_v}$ defines an element $c_{\rho|_{F_v}}\in H^1(F_v^{\mathrm{unr}},\ o_{\mathscr{D}}(1))$.

The subgroup $H^1_{\mathbf{n.o.f.}}(F_v^{\mathrm{unr}},\ W_2M_n^0)$ of $H^1(F_v^{\mathrm{unr}},\ W_2M_n^0)$ is defined as follows. Any element ρ' of $H^1(F_v^{\mathrm{unr}},\ W_2M_n^0)$ is regarded as an extension

$$0\longrightarrow\kappa_{\mathscr{D},v}\longrightarrow\rho'\longrightarrow\kappa_{\mathscr{D},v}(-1)\longrightarrow 0$$

over $\mathscr{O}_n[\epsilon]$, where the the extension class $c_{\rho'}$ in $H^1(F_v^{\mathrm{unr}},\ \mathscr{O}_n[\epsilon](1))$ lifts $c_{\rho|_{F_v}}\mod m_{\mathscr{D}}^n$. Then ρ' belongs to $H^1_{\mathbf{n.o.f.}}(F_v^{\mathrm{unr}},\ W_2M_n^0)$ if and only if $\mathrm{ev}_{\mathscr{O}_n[\epsilon]}(c_{\rho'})=0$ for the evaluation map (Definition 3.10)

Let $N_{\mathbf{n.o.f.},n}$ be the image of $H^1_{\mathbf{n.o.f.}}(F_v^{\mathrm{unr}},\ W_2M_n^0)$ in $H^1(F_v^{\mathrm{unr}},M_n^0)$.

$$H^1_{\mathbf{n.o.f.}}(F_v,\ M_n^0)=\ker(H^1(F_v,\ M_n^0)\to H^1(F_v^{\mathrm{unr}},\ M_n^0)/N_{\mathbf{n.o.f.},n}),$$

$$H^1_{\mathbf{n.o.f.}}(F_v,\ M_n)=H^1_{\mathbf{n.o.f.}}(F_v,\ M_n^0)\oplus H^1(F_v^{\mathrm{unr}}/F_v,\ \mathscr{O}_n).$$

Lemma 3.36. *Assume that* $\mathrm{def}_{\mathscr{D}}(v)=\mathbf{n.o.f.}$.

$$\mathrm{length}_{o_{\mathscr{D}}}H^1_{\mathbf{n.o.}}(F_v,\ M_n)/H^1_{\mathbf{n.o.f.}}(F_v,\ M_n)\le\mathrm{length}_{o_{\mathscr{D}}}o_{\mathscr{D}}/(\chi_{1,v}(\sigma)^2-\det\rho|_{F_v}(-1)(\sigma)).$$

Here σ is an element of G_{F_v} which lifts the geometric Frobenius element of $G_{k(v)}$.

Proof of Lemma 3.36. By the definition, $H^1_{\mathbf{n.o.}}(F_v,\ M_n)/H^1_{\mathbf{n.o.f.}}(F_v,\ M_n)$ is seen as a subgroup of $H^0(F_v^{\mathrm{unr}},\ \chi_{1,v}/\chi_{2,v}(-1))$ by

$$H^1_{\mathbf{n.o.}}(F_v,\ M_n)/H^1_{\mathbf{n.o.f.}}(F_v,\ M_n) \hookrightarrow N_{\mathbf{n.o.},n}/N_{\mathbf{n.o.f.},n} \simeq H^1(F_v^{\mathrm{unr}}, W_2M_n^0)/H^1_{\mathbf{n.o.f.}}(F_v^{\mathrm{unr}}, W_2M_n^0)$$
$$\overset{\mathrm{ev}}{\hookrightarrow} H^0(F_v^{\mathrm{unr}},\ \chi_{1,v}/\chi_{2,v}(-1)).$$

Since the image belongs to the G_{F_v}-invariant subspace, the claim follows. $\square$

Assume that $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{fl}$. Since M remains unchanged by a twist by a character, we may assume that the twist type κ_v is trivial.

Then we define $H^1_{\mathbf{fl}}(F_v,\ M_n)$ as the group of extensions

$$0 \longrightarrow L_n \longrightarrow E \longrightarrow L_n \longrightarrow 0$$

in $\mathscr{G}^{\mathscr{O}_n}_{F,[0,1]}$.

If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{f}$,

$$H^1_{\mathbf{f}}(F_v,\ M_n) = H^1_f(F_v,\ M_n) = H^1(F_v^{\mathrm{unr}}/F_v, M_n)$$

is the finite part. If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{u}$, then

$$H^1_{\mathbf{u}}(F_v,\ M_n) = H^1(F_v,\ M_n).$$

By the definition of $H^1_{\mathbf{f}}$ and $H^1_{\mathbf{u}}$, the following lemma follows easily.

Lemma 3.37. *Assume that* $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{f}$. *Then*

$$\mathrm{length}_{o_{\mathscr{D}}} H^1_{\mathbf{u}}(F_v,\ M_n)/H^1_{\mathbf{f}}(F_v,\ M_n) \leq \mathrm{length}_{o_{\mathscr{D}}} H^1(F_v,\ M_n)/H^1_f(F_v,\ M_n)$$
$$\leq \mathrm{length}_{o_{\mathscr{D}}} H^0(F_v,\ M_n(1)).$$

With these local groups defined for $v \in |F|_f$, we define the Selmer group for $\mathrm{ad}\,\rho$.

Definition 3.38. *Let ρ be a deformation of type $\mathscr{D}$ of $\bar{\rho}$.*

(1) *For any integer $n \geq 1$,*

$$H^1_{\mathscr{D}}(F,\ M_n) = \ker(H^1(F,\ M_n) \longrightarrow \bigoplus_{v\in|F|_f} H^1(F_v,\ M_n)/H^1_{\mathrm{def}_{\mathscr{D}}(v)}(F_v,\ M_n)).$$

(2) *The Selmer group is defined by*

$$\mathrm{Sel}_{\mathscr{D}}(F,\ M) = \cup_n \mathrm{Image}(H^1_{\mathscr{D}}(F,\ M_n) \to H^1(F,\ M \otimes_{o_{\mathscr{D}}} E_{\mathscr{D}}/o_{\mathscr{D}})).$$

As in [51], proposition 1.1, we have

Proposition 3.39. *Let ρ be a deformation of type $\mathscr{D}$ of $\bar{\rho}$ which corresponds to $f : R_{\mathscr{D}} \to o_{\mathscr{D}}$. Then*

$$\mathrm{Hom}_{o_{\mathscr{D}}}(\ker f/(\ker f)^2, E_{\mathscr{D}}/o_{\mathscr{D}}) \simeq \mathrm{Sel}_{\mathscr{D}}(F,\ M).$$

Remark 3.40. *Our Selmer group is related to the Selmer group in the sense of Bloch-Kato. As is discussed in* [9], *p.142, under the canonical map $H^1(F_v,\ M_n) \to H^1(F_v,\ M_{n+1})$ induced from the inclusion $M_n \hookrightarrow M_{n+1}$, the inverse image of $H^1_{\mathbf{fl}}(F_v,\ M_{n+1})$ is $H^1_{\mathbf{fl}}(F_v,\ M_n)$. We define $H^1_{\mathbf{fl}}(F_v,\ M)$ as the union of images of $H^1_{\mathbf{fl}}(F_v,\ M_n)$ in $H^1(F_v,\ M)$. This subspace is equal to the finite part $H^1_f(F_v, M)$ in the sense of Bloch-Kato: by Wiles' argument in* [51], *proposition 1.3, 1), $H^1_f(F_v,\ M) \subset H^1_{\mathbf{fl}}(F_v,\ M)$. Then the equality follows since $H^1_f(F_v,\ M)$ is divisible, and* $\mathrm{length}_{o_{\mathscr{D}}} H^1_f(F_v,\ M_1) \geq \mathrm{length}_{o_{\mathscr{D}}} H^1_{\mathbf{fl}}(F_v,\ M_1)$. *The last inequality follows from* [1], *proposition 1.9, and Theorem 3.20.*

4. Modular varieties and automorphic representations associated to quaternion algebras

In this section we briefly review modular varieties and automorphic representations associated to quaternion algebras. See [51], [23], [24] for further studies of Hecke algebras.

4.1. **Modular varieties associated to quaternion algebras.** Let F be a totally real number field of degree $[F:\mathbb{Q}]=d$. By $I_{F,\infty}$ we denote the set of all field embeddings $\iota : F \hookrightarrow \mathbb{R}$. $I_{F,\infty}$ is identified with the set $|F|_\infty$ of infinite places of F.

We take a quaternion algebra D which is central over F. By $D^\times$ we mean the multiplicative group of D, which is regarded as an algebraic group over F, and $G_D = \mathrm{Res}_{F/\mathbb{Q}} D^\times$ is the Weil restriction to $\mathbb{Q}$. Let Z be the center of G_D.

Let $I_D \subset I_{F,\infty}$ be the set of infinite places of F where D is split. We fix identifications

$$D \otimes_\iota \mathbb{R} \simeq M_2(\mathbb{R}) \ \text{ for } \iota \in I_D, \quad D \otimes_\iota \mathbb{R} \simeq \mathbb{H} \ \text{ for } \iota \in I_{F,\infty} \setminus I_D.$$

Here $\mathbb{H}$ is the Hamilton quaternion algebra. $G_D(\mathbb{R})$ is isomorphic to $\mathrm{GL}_2(\mathbb{R})^{I_D} \times (\mathbb{H}^\times)^{I_{F,\infty}\setminus I_D}$.

Let X_D be a $G_D(\mathbb{R})$-homogeneous space defined by

$$X_D = (\mathscr{H}^\pm)^{I_D}.$$

Here $\mathscr{H}^\pm = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ is the double half plane, and the $G_D(\mathbb{R})$-action is given by $G_D(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{R})^{I_D} \to \mathrm{PGL}_2(\mathbb{R})^{I_D} \simeq (\mathrm{Aut}\,\mathscr{H}^\pm)^{I_D}$ by taking the projection to $\mathrm{GL}_2(\mathbb{R})^{I_D}$. The complex dimension of X_D is equal to $\sharp I_D$, which we denote by q_D. We denote the stabilizer of $G_D(\mathbb{R})$ at $(\sqrt{-1},\dots,\sqrt{-1}) \in X_D$ by K_∞.

For a compact open subgroup $K \subset G_D(\mathbb{A}_{\mathbb{Q},f})$, the associated modular variety $S_K(G_D,X_D)(\mathbb{C})$ is defined by

$$S_K(G_D,X_D)(\mathbb{C}) = G_D(\mathbb{Q})\backslash G_D(\mathbb{A}_f) \times X_D/K = G_D(\mathbb{Q})\backslash G_D(\mathbb{A})/K \times K_\infty.$$

$S_K(G_D,X_D)(\mathbb{C})$ is viewed as the set of the $\mathbb{C}$-valued points of a $\mathbb{C}$-scheme $S_K(G_D,X_D)_\mathbb{C}$ of finite type.

For an inclusion of subgroups $K \to K'$, a natural projection

$$\pi_{K',K} : S_K(G_D,X_D)_\mathbb{C} \longrightarrow S_{K'}(G_D,X_D)_\mathbb{C}$$

is induced, and $\{S_K(G_D,X_D)_\mathbb{C}\}_{K\subset G_D(\mathbb{A}_{\mathbb{Q},f})}$ forms a projective system. $G_D(\mathbb{A}_{\mathbb{Q},f})$ acts on

$$S(G_D,X_D)_\mathbb{C} = \varprojlim_{K\subset G_D(\mathbb{A}_{\mathbb{Q},f})} S_K(G_D,X_D)_\mathbb{C}$$

by the right translation. The action of $g \in G_D(\mathbb{A}_{\mathbb{Q},f})$ is denoted by $R(g)$.

In this paper, we mainly consider the following two cases.

- $q_D = \sharp I_D = 1$: D is ramified at any infinite places except $\iota_0 \in I_{F,\infty}$.

In this case, $S_K(G_D,X_D)_\mathbb{C}$ is a Shimura curve, which is proper unless $F=\mathbb{Q}$ and $D = M_2(\mathbb{Q})$. By the theory of canonical models of Shimura [44], there is a model $S_K(G_D,X_D)_F$ canonically defined over F, $S_K(G_D,X_D)_F \times_{F,\iota_0} \mathrm{Spec}\,\mathbb{C} = S_K(G_D,X_D)_\mathbb{C}$. Since these models are canonical,

$$S(G_D,X_D)_F = \varprojlim_K S_K(G_D,X_D)_F$$

is defined over F, and hence the $G_D(\mathbb{A}_{\mathbb{Q},f})$-action is also defined over F.

- $q_D = \sharp I_D = 0$: D is a quaternion algebra over F which is ramified at all infinite places.

In this case, $S_K(G_D, X_D)_{\mathbb{C}}$ is a zero-dimensional scheme over $\mathbb{C}$, which we call the *Hida variety* associated to (G_D, X_D). A Hida variety is not a (non-connected) Shimura variety in the sense of Deligne (since it has a compact factor defined over $\mathbb{Q}$). The variety was first considered by Hida in his study of Hecke algebras for GL_2 [23].

4.2. **Equivariant sheaves on modular varieties.** In this subsection, D is a division algebra, and we denote $S_K(G_D, X_D)$ by S_K for short. For a prime ℓ and an ℓ-adic field E_λ, there is an E_λ-smooth sheaf $\bar{\mathscr{F}}^K_{(k,w),E_\lambda}$ on S_K (see [6] in the Shimura curve case, [23] for the Hida variety case). We discuss it here with a $\mathbb{Z}_\ell$-structure.

For a finite place $v|\ell$, $I_{F,v}$ is the set of field embeddings $F_v \hookrightarrow \bar{E}_\lambda$ over $\mathbb{Q}_\ell$, $I_{F,\ell} = \coprod_{v|\ell} I_{F,v}$. By an isomorphism $\bar{E}_\lambda \simeq \mathbb{C}$, $I_{F,\ell}$ is identified with $I_{F,\infty}$.

Definition 4.1. *Let (k, w) be a pair of an element k in $\mathbb{Z}^{I_{F,\infty}}$ and an integer w.*

1. *(k, w) is called an infinity type if $k = (k_\iota)_{\iota \in I_{F,\infty}}$ and w satisfy $k_\iota \geq 1$ and $k_\iota \equiv w$ $\mathrm{mod}\ 2$ for any $\iota \in I_{F,\infty}$.*
2. *An infinity type (k, w) is called a discrete type if $k_\iota \geq 2$ for any $\iota \in I_{F,\infty}$.*
3. *For $\iota \in I_{F,\infty}$, $k'_\iota = \frac{w-k_\iota}{2} + 1$.*

By the above identification, an infinity type (k, w) is also viewed as a pair $k \in \mathbb{Z}^{I_{E,\ell}}$ and $w \in \mathbb{Z}$.

We assume that D is split at all $v|\ell$, and choose an isomorphism $D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \prod_{v|\ell} M_2(F_v)$. By this isomorphism we regard $\prod_{v|\ell} \mathrm{GL}_2(o_{F_v})$ as a compact open subgroup of $G_D(\mathbb{Q}_\ell)$.

Let $V_{F_v} = F_v^{\oplus 2}$ be the standard representation of GL_{2,F_v} over F_v, $V_{o_{F_v}} = o_{F_v}^{\oplus 2}$ the standard lattice in V_{F_v} fixed by $\mathrm{GL}_2(o_{F_v})$. We take E_λ so that any $\iota : F_v \hookrightarrow \bar{E}_\lambda$ in $I_{F,\ell}$ factors through E_λ. For an infinity type (k, w) of discrete type, the representation

$$V_{(k,w),E_\lambda} = \bigotimes_{\iota : F_v \hookrightarrow E_\lambda, v|\ell} (\iota \det)^{-k'_\iota} \cdot \mathrm{Sym}^{k_\iota - 2}(V_{F_v} \otimes_\iota E_\lambda)^\vee$$

of $G_D(\mathbb{Q}_\ell)$ is defined over E_λ, and has an o_{E_λ}-lattice

$$V_{(k,w),o_{E_\lambda}} = \bigotimes_{\iota : F_v \hookrightarrow E_\lambda, v|\ell} (\iota \det)^{-k'_\iota} \cdot \mathrm{Sym}^{k_\iota - 2}(V_{o_{F_v}} \otimes_\iota o_{E_\lambda})^\vee,$$

which is stable under $\prod_{v|\ell} \mathrm{GL}_2(o_{F_v})$.

Definition 4.2. *Let K be a compact open subgroup of $G_D(\mathbb{A}_{\mathbb{Q},f})$. K is called small if for any compact open normal subgroup $K' \subset K$, the action of $(K \cap \overline{F^\times}) \cdot K' \backslash K$ on $S_{K'}$ induced by the right action of K is free.*

It is easily seen that if K is a small subgroup, then any compact open subgroup K' of K is again small. We discuss the smallness of a compact open subgroup in subsection 4.5.

For a small compact open subgroup $K = K_\ell \cdot K^\ell$ of $G_D(\mathbb{A}_{\mathbb{Q},f})$, let

$$\pi_\ell : \tilde{S}_\ell \longrightarrow S_K$$

be the Galois covering corresponding to $(K \cap \overline{F^\times})_\ell \backslash K_\ell$, where $(K \cap \overline{F^\times})_\ell$ is the image of $K \cap \overline{F^\times}$ by the projection $K \to K_\ell$.

If the action of $K \cap \overline{F^\times}$ on the representation $V_{(k,w),o_{E_\lambda}}$ is trivial, an o_{E_λ}-smooth sheaf $\bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}}$ on S_K is obtained from the covering π_ℓ and the representation $V_{(k,w),o_{E_\lambda}}$. Let g be an element of $G_D(\mathbb{A}_{\mathbb{Q},f})$. By the isomorphism $S_{gKg^{-1}} \overset{R(g)}{\xrightarrow{\sim}} S_K$ induced by the right $G_D(\mathbb{A}_{\mathbb{Q},f})$-action, $\bar{\mathscr{F}}^{gKg^{-1}}_{(k,w),o_{E_\lambda}} = R(g)^* \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}}$ is defined by the lattice $g_\ell V_{(k,w),o_{E_\lambda}}$, where g_ℓ is the ℓ-component of g.

For the right $G_D(\mathbb{A}_{\mathbb{Q},f})$-action on $S = \varprojlim_K S_K$, the E_λ-sheaf

$$\bar{\mathscr{F}}^K_{(k,w),\ E_\lambda} = \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}} \otimes_{o_{E_\lambda}} E_\lambda$$

forms a $G_D(\mathbb{A}_{\mathbb{Q},f})$-equivariant system of smooth sheaves as K varies by the construction. The o_{E_λ}-lattice structure is preserved by $G_D(\mathbb{A}^\ell_{\mathbb{Q},f})$-action.

When q_D is one, the sheaf $\bar{\mathscr{F}}^K_{(k,w)}$ is canonically defined over F by the theory of canonical models, which we denote by $\mathscr{F}^K_{(k,w)}$. This canonical F-structure gives a continuous G_F-action on the étale cohomology groups over $\bar{F}$.

The Betti-version of $\bar{\mathscr{F}}_{(k,w)}$ is discussed in [6], p.418–419. By the comparison theorem in étale cohomology, those two cohomology theories are canonically isomorphic, so we do not make any distinction unless otherwise stated.

Remark 4.3. *In the case of Shimura curves, $\mathscr{F}^K_{(k,w),E_\lambda}$ is pure of weight w.*

4.3. **Hecke correspondences and the duality formalism.** For a commutative ring R and a compact open subgroup K of $G_D(\mathbb{A}_{\mathbb{Q},f})$, let $H_{K,R} = H(G_D(\mathbb{A}_{\mathbb{Q},f}),\ K)_R$ be the convolution algebra formed by the compactly supported R-valued K-biinvariant functions on $G_D(\mathbb{A}_{\mathbb{Q},f})$. The right $G_D(\mathbb{A}_{\mathbb{Q},f})$-action on $S = \varprojlim_K S_K$ yields a left $G_D(\mathbb{A}_{\mathbb{Q},f})$-module structure on

$$H^*(S, \bar{\mathscr{F}}_{(k,w),E_\lambda}) \overset{def}{=} \varinjlim H^*(S_K, \bar{\mathscr{F}}^K_{(k,w),E_\lambda}).$$

At each finite level K, $H^*(S_K, \bar{\mathscr{F}}^K_{(k,w),E_\lambda})$ has a left action of the convolution algebra H_{K,E_λ}. In this subsection, we discuss this action in detail.

Definition 4.4. *Let K be a compact open subgroup K of $G_D(\mathbb{A}_{\mathbb{Q},f})$.*

(1) *K is $\mathbb{Q}$-factorizable if $K = \prod_q K_q$, $K_q \subset G_D(\mathbb{Q}_q)$. For a finite set Σ of primes , $K_\Sigma = \prod_{q\in\Sigma} K_q$, $K^\Sigma = \prod_{q\notin\Sigma} K_q$.*

(2) *K is F-factorizable if $K = \prod_{v\in|F|_f} K_v$, $K_v \subset D^\times(F_v)$ by the identification $G_D(\mathbb{A}_{\mathbb{Q},f}) = D^\times(\mathbb{A}_{F,f})$. For a finite set Σ of finite places of F, $K_\Sigma = \prod_{v\in\Sigma} K_v$, $K^\Sigma = \prod_{v\notin\Sigma} K_v$.*

(3) *For an F-factorizable compact open subgroup $K = \prod_v K_v$ of $G_D(\mathbb{A}_{\mathbb{Q},f})$, Σ_K is the set of finite places which satisfies the following property:*
 $v \notin \Sigma_K \Leftrightarrow D$ is split at v, and K_v is a maximal hyperspecial subgroup of $D^\times(F_v)$.

When $v \notin \Sigma_K$, K_v is isomorphic to $\mathrm{GL}_2(o_{F_v})$ by some group scheme isomorphism $D_v^\times \simeq \mathrm{GL}_{2.F_v}$ suitably taken.

For two compact open subgroups $K,\ K' \subset G_D(\mathbb{A}_{\mathbb{Q},f})$, $g \in G_D(\mathbb{A}_{\mathbb{Q},f})$ define an algebraic correspondence

$$[KgK'] : \quad S_K \xleftarrow{\mathrm{pr}_2} S_{K\cap gK'g^{-1}} \underset{\sim}{\xrightarrow{R(g^{-1})}} S_{g^{-1}Kg\cap K'} \xrightarrow{\mathrm{pr}_1} S_{K'},$$

where $\mathrm{pr}_1 = \pi_{g^{-1}Kg\cap K',K'}$ and $\mathrm{pr}_2 = \pi_{K\cap gK'g^{-1},K}$ correspond to the inclusion of groups, and the direction of the correspondence is from the second projection to the first projection, that is, from S_K to $S_{K'}$.

The algebraic correspondence induced by $K'gK$ from S_K to $S_{K'}$ is dual to $Kg^{-1}K'$ from $S_{K'}$ to S_K.

Since $\mathscr{F}_{(k,w),E_\lambda}$ is $G_D(\mathbb{A}_{\mathbb{Q},f})$-equivariant by the construction,

$$[KgK']^* : R\Gamma(S_{K'}, \bar{\mathscr{F}}^{K'}_{(k,w),E_\lambda}) \longrightarrow R\Gamma(S_K, \bar{\mathscr{F}}^K_{(k,w),E_\lambda})$$

is induced by

$$R\Gamma(S_{K'}, \bar{\mathscr{F}}^{K'}_{(k,w),E_\lambda}) \underset{\sim}{\xrightarrow{\mathrm{pr}_1^*}} R\Gamma(S_{g^{-1}Kg\cap K'}, \bar{\mathscr{F}}^{g^{-1}Kg\cap K'}_{(k,w),E_\lambda})$$

$$\xrightarrow{R(g^{-1})^*} R\Gamma(S_{K\cap gK'g^{-1}}, \bar{\mathscr{F}}^{K\cap gK'g^{-1}}_{(k,w),E_\lambda}) \longrightarrow R\Gamma(S_K, \bar{\mathscr{F}}^K_{(k,w),E_\lambda}).$$

We call $[KgK']^*$ the standard action of $[KgK']$. When $K = K'$, it is defined with the o_{E_λ}-lattice structure if $g_\ell^{-1}(V_{(k,w),o_{E_\lambda}}) \subset V_{(k,w),o_{E_\lambda}}$.

The action of the characteristic function χ_{KgK} of H_{K,E_λ} on $H^*(S_K, \bar{\mathscr{F}}^K_{E_\lambda})$ is the standard action $[KgK]^*$. Moreover, we have the action of $H(G_D(\mathbb{A}^\ell_{\mathbb{Q},f}),\ K^\ell)_{o_{E_\lambda}}$ on $H^*(S_K, \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}})$ for $K = K_\ell \cdot K^\ell$: For $g \in G_D(\mathbb{A}^\ell_{\mathbb{Q},f})$, $K^\ell g K^\ell$ acts by the standard action of $K\tilde{g}K$, $\tilde{g} = (1_{D(F_v)})_{v|\ell} \cdot g \in G_D(\mathbb{A}^\ell_{\mathbb{Q},f})$.

It is possible to lift the action of $H(G_D(\mathbb{A}^\ell_{\mathbb{Q},f}),\ K^\ell)_{o_{E_\lambda}}$ on the cohomology groups to $R\Gamma(S_K, \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}})$ as the following proposition shows:

Proposition 4.5. *There is a complex of left $H(G_D(\mathbb{A}^\ell_{\mathbb{Q},f}),\ K^\ell)_{o_{E_\lambda}}$-modules bounded below which represents $R\Gamma(S_K, \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}})$.*

Lemma 4.6. *Let $Y \overset{f_1,f_2}{\rightarrow} X$ be finite étale morphisms between schemes of finite type over $\mathbb{C}$, $\mathscr{F}$ an abelian sheaf on $X(\mathbb{C})$. We regard $X \overset{f_1}{\leftarrow} Y \overset{f_2}{\rightarrow} X$ as a correspondence, and assume that a cohomological correspondence $c : f_2^*\mathscr{F} \overset{\sim}{\rightarrow} f_1^*\mathscr{F}$ is given. Let $L^\cdot$ be the Godement's canonical resolution of $\mathscr{F}$. Then c induces a morphism of complexes $c^* : \Gamma(X(\mathbb{C}), L^\cdot) \to \Gamma(X(\mathbb{C}), L^\cdot)$ which gives the endmorphism of $R\Gamma(X(\mathbb{C}), \mathscr{F})$ in the derived category $D^+(X(\mathbb{C}))$ induced by c.*

Proof of Lemma 4.6. Since f_1 and f_2 are étale, they are local isomorphisms in the analytic category, and $f_i^*L^*$ is the canonical resolution of $f_i^*\mathscr{F}$ for $i = 1, 2$. Thus c induces an isomorphism of complexes $c^* : f_1^*L^\cdot \overset{\sim}{\rightarrow} f_2^*L^\cdot$. Since f_2 is finite étale, the trace map is defined for sheaves, thus $\mathrm{tr} : (f_2)_*f_2^*L^\cdot \to L^\cdot$ is defined. The composite of

$$\Gamma(X(\mathbb{C}), L^\cdot) \longrightarrow \Gamma(Y(\mathbb{C}), f_1^*L^\cdot)$$

$$\xrightarrow{\Gamma(c^*)} \Gamma(Y(\mathbb{C}), f_2^*L^\cdot) = \Gamma(X(\mathbb{C}), (f_2)_*f_2^*L^\cdot) \xrightarrow{\Gamma(\mathrm{tr})} \Gamma(X(\mathbb{C}), L^\cdot)$$

satisfies the desired property. □

Proof of Proposition 4.5. For simplicity, we work with the Betti realization described in [6], p.418–419. This is sufficient because the Betti and étale cohomologies are canonically isomorphic by the comparison theorem in étale cohomology.

We assume that the ℓ-component of g is 1. We have a morphism

$$\mathrm{pr}_1^*\,\bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}} = \bar{\mathscr{F}}^{g^{-1}Kg\cap K}_{(k,w),o_{E_\lambda}} \overset{R(g^{-1})^*}{\underset{\longrightarrow}{\sim}} \bar{\mathscr{F}}^{K\cap gKg^{-1}}_{(k,w),o_{E_\lambda}} = \mathrm{pr}_2^*\,\bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}}.$$

Let $L^\cdot$ be Godement's canonical resolution of $\bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}}$ on S_K. Since pr_1 and pr_2 are finite étale, Lemma 4.6 is applied, and the action of χ_{KgK} is actually defined on $\Gamma(S_K, L^\cdot)$. One checks that this action of χ_{KgK} extends to a left action of $H(G_D(\mathbb{A}^\ell_{\mathbb{Q},f}),\ K^\ell)_{o_{E_\lambda}}$, then $\Gamma(L^\cdot)$ defines an object of the derived category of $H(G_D(\mathbb{A}^\ell_{\mathbb{Q},f}),\ K^\ell)_{o_{E_\lambda}}$-modules bounded below which lifts $R\Gamma(S_K,\ \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}})$. □

For the relation between $H(G_D(\mathbb{A}^\ell_{\mathbb{Q},f}),\ K^\ell)_{o_{E_\lambda}}$-action and the Verdier duality, we have the following. By the definition of $\bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}}$,

$$(*) \qquad (\bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}})^\vee \overset{\sim}{\longrightarrow} \bar{\mathscr{F}}^K_{(k,-w),o_{E_\lambda}},$$

since $V_{(k,w),o_{E_\lambda}}$ satisfies $V^\vee_{(k,w),o_{E_\lambda}} \xrightarrow{\sim} V_{(k,-w),o_{E_\lambda}}$.

This gives a perfect pairing in the derived category of o_{E_λ}-modules

$$R\Gamma(S_K,\ \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}}) \otimes^{\mathbb{L}}_{o_{E_\lambda}} R\Gamma(S_K,\ \bar{\mathscr{F}}^K_{(k,-w),o_{E_\lambda}}) \longrightarrow o_{E_\lambda}(-q_D)[-2q_D]$$

by Poincaré duality. We need to know how Poincaré duality exchanges $H(G_D(\mathbb{A}^\ell_{\mathbb{Q},f}), K^\ell)_{o_{E_\lambda}}$-actions. By the isomorphism $(*)$, $(R(g)^*)^\vee$ is identified with $R(g^{-1})^*$. This implies that the standard action of KgK on $R\Gamma(S_K,\ \bar{\mathscr{F}}_{(k,w),o_{E_\lambda}})$ corresponds to the standard action of $Kg^{-1}K$ on $R\Gamma(S_K,\ \bar{\mathscr{F}}_{(k,-w),o_{E_\lambda}})$.

Proposition 4.7. *The standard action*

$$R\Gamma(S_{K'},\ \bar{\mathscr{F}}^{K'}_{(k,w),o_{E_\lambda}}) \stackrel{[KgK']^*}{\longrightarrow} R\Gamma(S_K,\ \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}})$$

induced by $[KgK']$ is dual to the standard action

$$R\Gamma(S_K,\ \bar{\mathscr{F}}^K_{(k,-w),o_{E_\lambda}})(q_D)[2q_D] \stackrel{[K'g^{-1}K]^*(q_D)[2q_D]}{\longrightarrow} R\Gamma(S_{K'},\ \bar{\mathscr{F}}^{K'}_{(k,-w),o_{E_\lambda}})(q_D)[2q_D]$$

by $[K'g^{-1}K]$.

We have two geometric actions of the convolution algebra, which we call the standard action and the dual action. The standard action of $[KgK]$ is $[KgK]^*$ we have already introduced. By the dual action of $[KgK]$ on $R\Gamma(S_K, \bar{\mathscr{F}}_{(k,w),o_{E_\lambda}})$, we mean the standard action of $[Kg^{-1}K]$. Proposition 4.7 implies that Poincaré duality exchanges the standard action to the dual action. The standard action (resp. dual action) is a left (resp. right) action.

For an F-factorizable compact open subgroup K and a finite place v of F such that D is split at v, choose a uniformizer p_v of F_v. Define $a(p_v)$ and $b(p_v) \in G_D(\mathbb{A}_{\mathbb{Q},f})$ as the elements having $\begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix}$, $\begin{pmatrix} p_v & 0 \\ 0 & p_v \end{pmatrix}$ as the v-component and the other components are 1, respectively.

As in §3.7, $U(p_v)$ and $U(p_v, p_v)$ are defined by $\chi_{Ka(p_v)K}$ and $\chi_{Kb(p_v)K}$. These operators are called the standard Hecke operators. They are independent of the choice of a uniformizer if K_v contains $K_0(m_v)$. When $K_v = \mathrm{GL}_2(o_{F_v})$, $U(p_v)$ and $U(p_v, p_v)$ are denoted by T_v and $T_{v,v}$.

4.4. **The reciprocity law for $S(G_D, X_D)$.** For a quaternion algebra D over F and an infinity type (k, w) of discrete type, let $\mathscr{A}^{\mathrm{disc}}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$ be the set of isomorphism classes of irreducible essentially square integrable representations of $G_D(\mathbb{A}_{\mathbb{Q}})$ of infinity type (k, w): for $\pi \in \mathscr{A}^{\mathrm{disc}}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$, π_∞ takes the following form

$$\pi_\infty = (\bigotimes_{\iota \in I_D} D_{k_\iota, w}) \otimes (\bigotimes_{\iota \in I_{F,\infty} \setminus I_D} \bar{D}_{(k_\iota, w)}),$$

where $\bar{D}_{(k_\iota,w)} = \mathrm{N}^{-k'_\iota}_{\mathbb{H}/\mathbb{R}} \cdot \mathrm{Sym}^{(k_\iota - 2)} V^\vee_{\mathrm{st}}$ is the irreducible representation of $\mathbb{H}^\times$ which corresponds to $D_{(k_\iota,w)}$ by the Jacquet-Langlands correspondence [27]. $\mathrm{N}_{\mathbb{H}/\mathbb{R}}$ is the reduced norm, V_{st} is the standard representation of $\mathrm{GL}_2(\mathbb{C})$, and we view $\mathrm{GL}_2(\mathbb{C})$-representation $\mathrm{Sym}^{(k_\iota-2)} V^\vee_{\mathrm{st}}$ as a representation of $\mathbb{H}^\times$. For a Hecke character $\chi : F^\times \backslash \mathbb{A}^\times_F \to \mathbb{C}^\times$, $\mathscr{A}^{\mathrm{disc}}_{(k,w),\chi}(G_D(\mathbb{A}_{\mathbb{Q}}))$ is the subset of $\mathscr{A}^{\mathrm{disc}}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$ consisting of the representations with the central character χ.

By the Jacquet-Langlands [27], Shimizu [40] correspondence, we have an injection

$$\mathrm{JL} : \mathscr{A}^{\mathrm{disc}}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}})) \longrightarrow \mathscr{A}^{\mathrm{disc}}_{(k,w)}(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2,F}(\mathbb{A}_{\mathbb{Q}})),$$

and $\mathscr{A}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$ (resp. $\mathscr{A}_{(k,w),\chi}(G_D(\mathbb{A}_{\mathbb{Q}}))$) is defined as the subset of $\mathscr{A}^{\mathrm{disc}}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$ (resp. $\mathscr{A}^{\mathrm{disc}}_{(k,w),\chi}(G_D)$) which correspond to cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$. The image of $\mathscr{A}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$ by JL consists of a cuspidal representation π which has an essentially square integrable component π_v at v where D is ramified. When D is a split quaternion algebra over F, we denote $\mathscr{A}^{\mathrm{disc}}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$ (resp. $\mathscr{A}^{\mathrm{disc}}_{(k,w),\chi}(G_D(\mathbb{A}_{\mathbb{Q}}))$, resp. $\mathscr{A}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$, resp. $\mathscr{A}_{(k,w),\chi}(G_D(\mathbb{A}_{\mathbb{Q}}))$) by $\mathscr{A}^{\mathrm{disc}}_{F,(k,w)}$ (resp. $\mathscr{A}^{\mathrm{disc}}_{F,(k,w),\chi}$, resp. $\mathscr{A}_{F,(k,w)}$, resp. $\mathscr{A}_{F,(k,w),\chi}$).

First consider the Shimura curve case. For an infinity type (k,w) of discrete type, the decomposition of the étale cohomology groups as $G_F \times H(G_D(\mathbb{A}_{\mathbb{Q}}),K)$-bimodules is given by

$$H^1_{\text{ét}}(S_{K,\bar{F}},\ \bar{\mathscr{F}}^K_{(k,w),\bar{E}_\lambda}) \simeq \bigoplus_{\pi\in\mathscr{A}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))} \rho_{\pi,\bar{E}_\lambda} \otimes_{\bar{E}_\lambda} \pi_f^K,$$

and

$$H^0_{\text{ét}}(S_{K,\bar{F}},\ \bar{\mathscr{F}}^K_{(k,w),\bar{E}_\lambda}) \oplus H^2_{\text{ét}}(S_{K,\bar{F}},\ \bar{\mathscr{F}}^K_{(k,w),\bar{E}_\lambda}) \simeq \oplus_{\pi\in\mathscr{A}^c_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))}\rho_{\pi,\bar{E}_\lambda} \otimes_{\bar{E}_\lambda} \pi_f^K.$$

Here $\mathscr{A}^c_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}})) = \mathscr{A}^{\mathrm{disc}}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}})) \setminus \mathscr{A}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$. For $\pi \in \mathscr{A}_{(k,w)}(G_D)$, $\rho_{\pi,\bar{E}_\lambda} : G_F \to \mathrm{GL}_2(\bar{E}_\lambda)$ is the two dimensional irreducible ℓ-adic representation associated to π ([33],[6]), and we view the finite part π_f of π is defined over $\bar{E}_\lambda$ by the identification $\bar{E}_\lambda \simeq \mathbb{C}$. Note that $\mathscr{A}^c_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$ is non-empty if and only if $k = (2,\dots,2)$, and $\mathscr{A}^c_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$ consists of one dimensional representations which factor through the reduced norm $D^\times(\mathbb{A}_F) \overset{\mathrm{N}_{D/F}}{\to} \mathbb{A}_F^\times \overset{\chi}{\to} \mathbb{C}^\times$. In this case, χ is an algebraic Hecke character of weight w, and $\rho_{\pi,\bar{E}_\lambda}$ is $\rho_{\chi,\bar{E}_\lambda} \oplus \rho_{\chi,\bar{E}_\lambda}(-1)$, where $\rho_{\chi,\bar{E}_\lambda} : G_F \to \bar{E}_\lambda^\times$ is the G_F-representation attached to the algebraic Hecke character χ.

In the case of Hida varieties, there are no natural Galois actions, still there is a decomposition

$$H^0_{\text{ét}}(S_{K,\mathbb{C}},\ \bar{\mathscr{F}}^K_{(k,w),\bar{E}_\lambda}) \simeq I^K_{(k,w),E_\lambda} \oplus (\bigoplus_{\pi\in\mathscr{A}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))} \pi_f^K)$$

as a $H(G_D(\mathbb{A}_{\mathbb{Q}}),K)$-module. Here $I^K_{(k,w),\bar{E}_\lambda}$ is the subspace of $H^0_{\text{ét}}(S_{K,\mathbb{C}},\ \bar{\mathscr{F}}^K_{(k,w),\bar{E}_\lambda})$ consisting of section f such that the lift of f to $D^\times(\mathbb{A}_{F,f})$ factors through $D^\times(\mathbb{A}_{F,f}) \overset{\mathbb{N}_{D/F}}{\to} (\mathbb{A}_{F,f})^\times$. $I^K_{(k,w),\bar{E}_\lambda}$ is isomorphic to $\bigoplus_{\pi\in\mathscr{A}^c_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))} {\pi_f}^K$.

In the both cases (especially when $q_D = 0$), the Galois representation $\rho_{\pi,\bar{E}_\lambda}$ attached to $\pi \in \mathscr{A}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$ exists, and is isomorphic to $\rho_{\mathrm{JL}(\pi),\bar{E}_\lambda}$ attached to the cuspidal representation $\mathrm{JL}(\pi)$ of $\mathrm{GL}_2(\mathbb{A}_F)$.

Remark 4.8. *There is an action of the field automorphisms* $\mathrm{Aut}\,\bar{E}_\lambda$ *on* $\mathscr{A}_{(k,w)}(G_D)$*: for a representation* $\pi \in \mathscr{A}_{(k,w)}(G_D)$ *and any element* $\tau \in \mathrm{Aut}\,\bar{E}_\lambda$*, let* π_f^τ *be the twist of* π_f *by* τ*:* $\pi_f^\tau : G_D(\mathbb{A}_{\mathbb{Q},f}) \overset{\pi_f}{\to} \mathrm{Aut}_{\bar{E}_\lambda} V \simeq \mathrm{Aut}_{\bar{E}_\lambda} V \otimes_{\bar{E}_\lambda,\tau} \bar{E}_\lambda$*. Then there is a representation* $\pi^\tau \in \mathscr{A}_{(k,w)}(G_D)$ *such that* $(\pi^\tau)_f \simeq \pi_f^\tau$*.* π^τ *is unique up to isomorphisms.*

Since the $\mathrm{Aut}\,\bar{E}_\lambda$*-orbit of* π *is a finite set, for the stabilizer* H_π *of the isomorphism class of* π*,*

$$E_\pi = \bar{E}_\lambda^{H_\pi}$$

is a number field of finite degree over $\mathbb{Q}$*, which we call the field of definition of* π*. When a subfield* E *of* $\bar{E}_\lambda$ *contains* E_π*, we say that* π *is defined over* E*. By the strong multiplicity one theorem for* GL_2*,* $\pi^\tau \simeq \pi$ *if and only if* $\pi_v^\tau \simeq \pi_v$ *for almost all* v*. In particular* π_v *is defined over* E *if and only if* π_v *is defined over* E *for almost all* v*.*

4.5. **Smallness of compact open subgroups.**

Proposition 4.9. *Let K be an F-facrtorizable compact open subgroup of $G_D(\mathbb{A}_{\mathbb{Q},f})$, y a finite place of F, S a finite set of finite places of F. Assume the following conditions:*

(1) *$y \notin S$, $\Sigma_K \cap (\{y\} \cup S) = \emptyset$, and any element in S and y do not divide 2.*
(2) *The map $o_F^\times/(o_F^\times)^2 \to \prod_{v\in S} k(v)/(k(v))^2$ is injective.*
(3) *$G_D^{\mathrm{der}}(\mathbb{Q}) \cap g^{-1}(K_{11}(m_y)\cdot K^y)g$ is torsion free for any $g \in G_D(\mathbb{A}_{\mathbb{Q},f})$.*

For $u \in S$, define a compact open subgroup U_u of $D^\times(F_u)$ by

$$U_u = \{\alpha \in K_u, \mathrm{N}_{D_{F_u}/F_u}(\alpha) \in (o_{F_u}^\times)^2\}.$$

Then $K(y,S) = (K_{11}(m_y)\cap K_y)\cdot \prod_{u\in S} U_u \cdot K^{\{y\}\cup S}$ is a small subgroup of $G_D(\mathbb{A}_{\mathbb{Q},f})$.

To show the proposition, it suffices to show the following lemma.

Lemma 4.10. *For an open subgroup K' of $K(y,S)$ and an element k of $K(y,S)$, assume that k normalizes K'. If the right action of k on $S_{K'}$ admits a fixed point, there is a unit $\delta \in o_F{}^\times$ such that $\delta^{-1}k \in K'$.*

Proof of Lemma 4.10. We may assume that any local component K_v of K at a finite place v is a maximal compact open subgroup of $D^\times(F_v)$, and $K_y = K_{11}(m_y)$.

Assume that k fixes the double coset $G_D(\mathbb{Q})x(K'\cdot K_\infty)$ defined by $x \in G_D(\mathbb{A}_{\mathbb{Q}})$. There are elements $\gamma \in G_D(\mathbb{Q})$ and $k' \in K'$ such that

$$x\cdot k = \gamma \cdot x \cdot k' \tag{$*$}$$

holds. By taking the reduced norm of $(*)$, $\epsilon = \mathrm{N}_{D/F}(k')\cdot \mathrm{N}_{D/F}(k)^{-1}$ for $\epsilon = \mathrm{N}_{D/F}(\gamma)$. ϵ is a unit of F since it belongs to $\widehat{o_F^\times}$. At $u \in S$, ϵ belongs to $(o_{F_u}^\times)^2$ by the definition of $K(y,S)$. By Proposition 4.9 (2), this implies that $\epsilon \in (o_F^\times)^2$. We take $\delta \in o_F^\times$ such that $\delta^2 = \epsilon$, and $\delta \mod m_{F_y} = 1$. $\delta \in \tilde{K} = K(y,S)$, and $\tilde{\gamma} = \gamma/\delta$ belongs to $G_D^{\mathrm{der}}(\mathbb{Q})$.

Equation $(*)$ reduces to

$$x\cdot \tilde{k} = \tilde{\gamma}\cdot x, \tag{$\dagger$}$$

where $\tilde{k} = k/(\delta\cdot k') \in \tilde{K}$.

When $q_D = 0$, $(\dagger)$ implies that $\tilde{\gamma}$ fixes a $\mathbb{Z}$-lattice in some faithful $\mathbb{Q}$-representation of G_D^{der} since it is contained in $x\tilde{K}x^{-1}$. Thus $\tilde{\gamma}$ is an element of finite order, because it is contained in the compact group $G_D^{\mathrm{der}}(\mathbb{R})$. By our assumption 4.9 (3), $\tilde{\gamma} = 1$, and $\tilde{k} = 1$ follows from $(\dagger)$. This implies the freeness of the action on $S_{K'}$.

When $q_D = 1$, $(\dagger)$ implies that the finite part x_f of x satisfies $\tilde{\gamma} \in x_f\tilde{K}x_f^{-1}$, and the class of the infinite part $x_\infty K_\infty \in X_D$ is fixed by $\tilde{\gamma}$ from the left. By Proposition 4.9 (3), the action of $G_D^{\mathrm{der}}(\mathbb{Q})\cap x_f(K_{11}(m_y)\cdot K^y)x_f^{-1}$ on X_D is free, which implies that $\tilde{\gamma} = 1$. It follows that $\tilde{k} = 1$, and the claim is shown. □

As for the existence of a nice pair (y,S) as in Proposition 4.9, we have the following lemma.

Lemma 4.11. *Let K be an F-factorizable compact open subgroup of $G_D(\mathbb{A}_{\mathbb{Q},f})$.*

(1) *For any finite set P_0 of finite places of F which contains $\Sigma_K \cup \{u; u|2\}$, there is a finite set S of finite places that is disjoint from P_0, and the condition (2) of Proposition 4.9 is satisfied for S.*
(2) *There is an integer $a_F \geq 1$, which depends only on F, with the following property: for any finite place $y \notin \Sigma_K$ and $q_y \geq a_F$, the condition (3) of 4.9 is satisfied for y.*

Proof of Lemma 4.11. To construct S, let F' be the Galois extension of F defined by $F' = F(\sqrt{\epsilon},\ \epsilon \in o_F^\times)$. The Galois group $G' = \mathrm{Gal}(F'/F)$ is an abelian group of type $(2, \dots, 2)$, which we view as an $\mathbb{F}_2$-vector space. Let $\{\sigma_j\}_{j \in J}$ be a basis of G' over $\mathbb{F}_2$. We take S so that the following two conditions are satisfied:

- $S \cap P_0 = \emptyset$,
- For any $j \in J$, there is an element s_j in S such that F' is unramified at s_j, and the geometric Frobenius element Fr_{s_j} at s_j is mapped to σ_j.

The existence of S is guaranteed by the Chebotarev density theorem, and Proposition 4.9 (2) is satisfied for this choice of S.

For (2), the existence of a_F is proved in [23], lemma 7.1 (we take a_F so that $a_F > 2^d$, which excludes the possibility that y divides 2). □

Remark 4.12. *The notations are as in Lemma 4.11. Let $\mathscr{E}$ be the kernel of $\mathrm{N}_{F/\mathbb{Q}} : o_F^\times \to \mathbb{Z}^\times$. If we add a finite place s, which is split in $F(\sqrt{\epsilon}, \epsilon \in \mathscr{E})$ but does not split in $F(\sqrt{-1})$, to S, $\mathscr{F}^{K'}_{(k,w),E_\lambda}$ is defined on $S_{K'}$ for any compact open subgroup K' of $K(y, S)$.*

For an F-factorizable compact open subgroup K of $G_D(\mathbb{A}_{\mathbb{Q},f})$, assume that y satisfies the condition (1) of Proposition 4.9, and the y-component K_y of K is contained in $K_{11}(m_y)$. By Lemma 4.11, we take an auxiliary set S which is disjoint from $\Sigma_K \cup \{u : u|2\}$ and satisfies the condition (2) of Proposition 4.9 (cf. Remark 4.12).

When $\ell \geq 3$, for any integer q and a finite o_λ-algebra R, the cohomology group $H^q_{\mathrm{stack}}(S_K\ \bar{\mathscr{F}}^K_{(k,w),R})$ of degree q is defined as the K-invariant part $H^q(S_{K(y,S)}\ \bar{\mathscr{F}}^{K(y,S)}_{(k,w),R})^K$ of $H^q(S_{K(y,S)}\ \bar{\mathscr{F}}^{K(y,S)}_{(k,w),R})$.

Since $K(y, S)\backslash K$ is an abelian group of type $(2, \dots, 2)$ and $\ell \geq 3$, $H^q_{\mathrm{stack}}(S_K\ \bar{\mathscr{F}}^K_{(k,w),R})$ is an R-direct summand of $H^q(S_{K(y,S)}\ \bar{\mathscr{F}}^{K(y,S)}_{(k,w),R})$. This property is usually suffcient to make an analysis on $S_{K(y,S)}$. In particular exact sequences are preserved.

Remark 4.13. *As the notation suggests, $H^q_{\mathrm{stack}}(S_K)$ is canonically isomorphic to the (étale) cohomology group of S_K when S_K is regarded as a Deligne-Mumford stack, and is independent of the choice of an auxiliary set S.*

5. Universal exactness of cohomology sequences

We study the exactness of a homomorphism defined by degeneracy maps on cohomology groups. In particular we show the sequences in consideration are universally exact, that is, the sequences remain exact under any extensions of scalars. This is a basic tool in the study of cohomological congruences, in particular in the theory of congruence modules.

In the elliptic modular case and the subgroup is $K_0(v)$, Ribet calls the universal injectivity "Ihara's Lemma".

Throughout this section, D is a division algebra unless otherwise stated.

5.1. **Modules of residual type.** Let K be an F-factorizable compact open subgroup of $G_D(\mathbb{A}_{\mathbb{Q},f})$. For a finite set of finite places Σ which contains Σ_K and the places dividing ℓ, let $T_\Sigma = H(D^\times(\mathbb{A}_f^\Sigma),\ K^\Sigma)_{o_{E_\lambda}}$ be the convolution algebra consting of o_{E_λ}-valued functions. By the assumption on Σ, T_Σ is commutative.

Definition 5.1. *Consider the category $\mathscr{C}_{T_\Sigma}$ of T_Σ-modules which are finitely generated as o_{E_λ}-modules.*

(1) *We call an object N in $\mathscr{C}_{T_\Sigma}$ residual type if any constituent N' of $N/\lambda N$ satisfies the relation*
$$[T_v]^2 = [T_{v,v}](1 + q_v)^2$$
on N' for any $v \notin \Sigma$.

(2) *A maximal ideal m of T_Σ is of residual type if T_Σ/m is of residual type.*

By $\mathscr{C}_{\mathrm{res}}$ we denote the subcategory of $\mathscr{C}_{T_\Sigma}$ consisting of the T_Σ-modules of residual type.

$\mathscr{C}_{\mathrm{res}}$ is a Serre subcategory of $\mathscr{C}_{T_\Sigma}$, and is stable under the dual action of T_Σ. By $\bar{\mathscr{C}}_{T_\Sigma}$, we mean the quotient category of $\mathscr{C}_{T_\Sigma}$ by $\mathscr{C}_{\mathrm{res}}$.

A typical example of modules of residual type is obtained by a one dimensional representation $\pi : D^\times(\mathbb{A}_f^\Sigma) \to E_\lambda^\times$ which factors through the reduced norm $\mathrm{N}_{D/F}$. The induced T_Σ-action gives a module of residual type.

For a Galois representation $\bar\rho : G_\Sigma \to \mathrm{GL}_2(k_\lambda)$, the maximal ideal $m_{\bar\rho}$ of T_Σ associated to $\bar\rho$ is the kernel of o_{E_λ}-algebra homomorphism $f_{\bar\rho} : T_\Sigma \to k_\lambda$ such that $f_{\bar\rho}(T_v) = \mathrm{trace}\bar\rho(\mathrm{Fr}_v)$, $f_{\bar\rho}(T_{v,v}) = q_v^{-1}\det\bar\rho(\mathrm{Fr}_v)$. The following proposition shows that the maximal ideals of T_Σ of residual type coming from Galois representations correspond to very special reducible representations.

Proposition 5.2. *For a continuous representation $\bar\rho : G_\Sigma \to \mathrm{GL}_2(\bar k_\lambda)$, the maximal ideal $m_{\bar\rho}$ corresponding to $\bar\rho$ is of residual type if and only if $\bar\rho^{ß}$ satisfies*

$$\bar\rho^{ß} \simeq \bar\chi \oplus \bar\chi(-1)$$

for some one dimensional character $\bar\chi : G_\Sigma \to \bar k_\lambda^\times$ over $\bar k_\lambda$.

A proof is found in [15], Proposition 3.6.

Remark 5.3. (1) *The notion of modules of residual type is stronger than the notion of Eisenstein modules in* [12], *and is introduced in* [15] *(there it is called of ω-type).*

(2) *Let m be a maximal ideal of T_Σ which is not of residual type. If a sequence*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

of T_Σ-modules is exact in $\bar{\mathscr{C}}_{T_\Sigma}$, then the sequence

$$0 \longrightarrow (M_1)_m \longrightarrow (M_2)_m \longrightarrow (M_3)_m \longrightarrow 0$$

localized at m is exact.

5.2. **Universality under scalar extensions.** For a finite o_{E_λ}-algebra R, we denote $\bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}} \otimes_{o_{E_\lambda}} R$ by $\bar{\mathscr{F}}^K_{(k,w),R}$.

Lemma 5.4. *Assume that D is a division algebra which defines a Shimura curve, K is an F-factorizable compact open subgroup of $D^\times(\mathbb{A}_{F,f})$, and Σ is a finite set of finite places of F that contains $\Sigma_K \cup \{v|\ell\}$.*

For any integer $\alpha \geq 1$ and any $D^\times(\mathbb{A}^\Sigma_{F,f})$-equivariant k_λ smooth subquotient $\mathscr{F}$ of $(\bar{\mathscr{F}}^K_{(k,w),k_\lambda})^{\oplus\alpha}$, $H^q(S_K, \mathscr{F})$ is of residual type for $q = 0, 2$.

Proof of Lemma 5.4. First we prove the claim for $q = 0$. For an inclusion of two groups $K' \hookrightarrow K$, $H^0(S_K, \mathscr{F})$ is a subspace of $H^0(S_{K'}, \mathscr{F}_{K'})$ since $\pi_{K',K}$ is surjective. Here $\mathscr{F}_{K'}$ is the pullback of $\mathscr{F}$ to $S_{K'}$. So we may assume that $K_u \subset K(m_{F_u})$ for $u|\ell$. By the definition of $\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}}$, $\bar{\mathscr{F}}_{(k,w),k_\lambda}$ is trivialized in $D^\times(\mathbb{A}^\Sigma_{F,f})$-equivariant way on S_K. This implies that any subquotient of $\mathscr{F}$ is isomorphic to k_λ as a $D^\times(\mathbb{A}^\Sigma_{F,f})$-equivariant sheaf. If $F = k_\lambda$, the claim follows from the isomorphism $\pi_0(S_K) \simeq \pi_0(F^\times\backslash\mathbb{A}^\times_{F,f}/\mathrm{N}_{D/F}K)$, by taking a finite set of finite places Σ which contains Σ_K, and all places dividing ℓ. By the induction on the k_λ-rank of F, the claim is shown in this case.

For $q = 2$, this follows from the case of $q = 0$ by Poincaré duality. □

Proposition 5.5. *Assume that D is a division algebra with $q_D \leq 1$. For any finite o_{E_λ}-algebra R, $H^{q_D}(S_K, \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}})$ is λ-torsion free in $\bar{\mathscr{C}}_{T_\Sigma}$, and*

$$H^{q_D}(S_K, \bar{\mathscr{F}}^K_{(k,w),R}) = H^{q_D}(S_K, \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}}) \otimes_{o_{E_\lambda}} R$$

holds in $\bar{\mathscr{C}}_{T_\Sigma}$.

Proof of Proposition 5.5. The claim is clear for definite quaternion algebras. So we may assume that $q_D = 1$, and D defines a Shimura curve. In the following exact sequence

$$H^0(S_K,\ \bar{\mathscr{F}}^K_{(k,w),k_\lambda}) \longrightarrow H^1(S_K,\ \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}}) \xrightarrow{\lambda} H^1(S_K,\ \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}})$$
$$\longrightarrow H^1(S_K,\ \bar{\mathscr{F}}^K_{(k,w),k_\lambda}) \longrightarrow H^2(S_K,\ \bar{\mathscr{F}}^K_{(k,w),k_\lambda}),$$

$H^q(S_K,\ \bar{\mathscr{F}}^K_{(k,w),k_\lambda})$ for $q = 0, 2$ are of residual type by Lemma 5.4. Thus $H^1(S_K,\ \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}})$ is λ-torsion free, and $H^1(S_K,\ \bar{\mathscr{F}}^K_{(k,w),o_{E_\lambda}}) \otimes_{o_{E_\lambda}} k_\lambda = H^1(S_K,\ \bar{\mathscr{F}}^K_{(k,w),k_\lambda})$ in $\bar{\mathscr{C}}_{T_\Sigma}$. For general coefficient R, we reduce to the case when R has a finite length, and in that case it follows from an induction on length$R$. $\square$

Proposition 5.6. *For a finite complex $L = (L^\cdot, d^\cdot)$ of finite free o_{E_λ}-modules, define L_R by $L_R = (L^\cdot \otimes_{o_{E_\lambda}} R, d^\cdot \otimes \mathrm{id}_R)$ for any finite o_{E_λ}-algebra R. Then the following properties are equivalent:*

(1) *For any finite o_{E_λ}-algebra R, L_R is exact.*
(2) *L_{k_λ} is exact.*
(3) *L is exact, and the image of $d^i : L^i \to L^{i+1}$ in L^{i+1} is an o_{E_λ}-direct summand for any i.*

The verification is left to the reader.

Definition 5.7. *Let L be a finite complex of finite free o_{E_λ}-modules. If L satisfies the equivalent conditions in Proposition 5.6, we say L is universally exact. For an o_{E_λ}-homomorphism $f : L_0 \to L_1$ between finite free o_{E_λ}-modules, f is called universally injective if $[0 \to L_0 \xrightarrow{f} L_1]$ is universally exact.*

5.3. **Cohomological universal injectivity.**

Proposition 5.8. *Let D be a division quaternion algebra with $q_D \leq 1$. For an F-factorizable small compact open subgroup K of $G_D(\mathbb{A}_{\mathbb{Q},f})$ and a compact open subgroup K' of K,*

$$H^{q_D}(S_K, \bar{\mathscr{F}}^K_{(k,w),o_\lambda}) \xrightarrow{\pi^*_{K',K}} H^{q_D}(S_{K'}, \bar{\mathscr{F}}^{K'}_{(k,w),o_\lambda})$$

is universally injective up to modules of residual type.

Proof of Proposition 5.8. By Proposition 5.5 and 5.6, it is sufficient to show the kernel of

$$H^{q_D}(S_K, \bar{\mathscr{F}}^K_{(k,w),k_\lambda}) \xrightarrow{\pi^*_{K',K}} H^{q_D}(S_{K'}, \bar{\mathscr{F}}^{K'}_{(k,w),k_\lambda})$$

is a module of residual type. If $q_D = 0$, this is clear. So we may assume that $q_D = 1$. Moreover, by replacing K' by an F-factorizable open subgroup, it is enough to consider the case when K' is a normal subgroup of K. $\pi^*_{K',K}$ is an étale torsor under $G = (\overline{F^\times} \cap K) \cdot K' \backslash K$, so we have the following exact sequence

$$H^1(G, H^0(S_{K'}, \bar{\mathscr{F}}^{K'}_{(k,w),k_\lambda})) \longrightarrow H^1(S_K, \bar{\mathscr{F}}^K_{(k,w),k_\lambda}) \xrightarrow{\pi^*_{K',K}} H^1(S_{K'}, \bar{\mathscr{F}}^{K'}_{(k,w),k_\lambda}).$$

For $\Sigma = \Sigma_{K'} \cup \{v|\ell\}$, the action of T_Σ on $N = H^0(S_{K'}, \bar{\mathscr{F}}^{K'}_{(k,w),k_\lambda})$ commutes with the G-action on N, and is of residual type by Lemma 5.4. So $H^1(G, N)$ is also of residual type. □

Let D be a division quaternion algebra with $q_D \leq 1$. For an F-factorizable compact open subgroup K of $G_D(\mathbb{A}_{\mathbb{Q},f})$, assume that D is split at v, and the v-component of K is $\mathrm{GL}_2(o_{F_v})' = \ker(\mathrm{GL}_2(o_{F_v}) \overset{\det}{\to} o_{F_v}^\times \to k(v)^\times)$.

Let $\mathrm{pr}_{i,v} : S_{K_0(v)\cap K} \to S_K$ $(i = 1, 2)$ be two degeneracy maps defined as follows. $\mathrm{pr}_2 = \pi_{K_0(v)\cap K,K}$ is the canonical projection corresponding to the inclusion $K \cap K_0(v) \subset K$, $\mathrm{pr}_1 = \pi_{a(p_v)^{-1}(K_0(v)\cap K)a(p_v),K} \circ R(a(p_v)^{-1})$ is the projection twisted by the conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix}$ at v, that is,

$$K_v \cap K_0(m_v) \longrightarrow K_v \cap \begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix}^{-1} K_0(m_{F_v}) \begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix} = K_v \cap K^{\mathrm{op}}(v) \subset K_v.$$

Here $K^{\mathrm{op}}(v) = K^{\mathrm{op}}(m_{F_v}) \cdot K^v$, and $K^{\mathrm{op}}(m_{F_v}) = \{{}^t g;\ g \in K_0(m_{F_v})\}$.

By the definition, $S_K \overset{\mathrm{pr}_1}{\leftarrow} S_{K\cap K_0(v)} \overset{\mathrm{pr}_2}{\to} S_K$ is equal to $[Ka(p_v)K]$ as a correspondence.

Consider the map

$$H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w),o_{E_\lambda}})^{\oplus 2} \overset{\mathrm{pr}_1^* + \mathrm{pr}_2^*}{\longrightarrow} H^{q_D}(S_{K_0(v)\cap K},\ \bar{\mathscr{F}}_{(k,w),o_{E_\lambda}}).$$

Later in §10, we need a universal injectivity (Ihara's Lemma) in the calculation of congruence modules. We state it as a hypothesis here.

Hypothesis 5.9 (Cohomological universal injectivity)**.** *Let D be a quaternion algebra over F with $q_D \leq 1$ which is split at a finite place v. Assume that K is an F-factorizable compact open subgroup of $G_D(\mathbb{A}_{\mathbb{Q},f})$, with the v-component $K_v = \mathrm{GL}_2(o_{F_v})'$. For a discrete infinite type (k, w), if $v|\ell$, we further assume that $k_\iota = 2$ and $w = 0$ for any $\iota \in I_{F,v}$.*

Then

$$H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w),o_{E_\lambda}})^{\oplus 2} \overset{\mathrm{pr}_1^* + \mathrm{pr}_2^*}{\longrightarrow} H^{q_D}(S_{K\cap K_0(v)},\ \bar{\mathscr{F}}_{(k,w),o_{E_\lambda}})$$

is universally injective.

As for the validity of Hypothesis 5.9, we prove the following theorem.

Theorem 5.10. *Hypothesis 5.9 is true when $q_D = 0$.*

Before beginning the proof of Theorem 5.10, let us recall the following general fact.

Lemma 5.11. *Assume that $f_1 : X \to X_1$, $f_2 : X \to X_2$ are two surjective maps between finite sets.*

(1)

$$0 \to \ker(f_1^* + f_2^*) \longrightarrow H^0(X_1,\ o_{E_\lambda}) \oplus H^0(X_2,\ o_{E_\lambda}) \overset{f_1^*+f_2^*}{\longrightarrow} H^0(X,\ o_{E_\lambda})$$

is universally exact.

(2) $\ker(f_1^* + f_2^*)$ *is identified with $H^0(X_1 \amalg_X X_2,\ o_{E_\lambda})$. Here $X_1 \amalg_X X_2$ is the coproduct with respect to f_1 and f_2.*

A proof of (1) is found in [46], lemma 4, case 1. We give a proof based on the theorem of van Kampen.

Proof of Lemma 5.11. For any o_{E_λ}-algebra R, define a complex L_R of R-modules by

$$L_R = [H^0(X_1,\ R) \oplus H^0(X_2,\ R) \overset{f_1^*+f_2^*}{\longrightarrow} H^0(X,\ R)].$$

Here the components are placed at degree 0 and 1. The formation $R \mapsto L_R$ commutes with any extension of scalars $R \to R'$: $L_R \otimes_R R' = L_{R'}$ holds.

It is easily seen that $H^0(L_R)$ is identified with the cohomology group of the coproduct $H^0(X_1 \amalg_X X_2,\ R)$. The identification is

$$H^0(X_1 \amalg_X X_2,\ R) \xrightarrow{(g_1^*, -g_2^*)} H^0(X,\ R)^{\oplus 2}$$

for the maps defined by the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "f_2"'] & X_1 \arrow[d, "g_1"] \\ X_2 \arrow[r, "g_2"] & X_1 \amalg_X X_2 \end{tikzcd}$$

It follows that the formation $R \mapsto H^0(L_R)$ also commutes with any extension of scalars. So $H^1(L_R)$ must satisfy the same property since $L_{o_{E_\lambda}}$ is a perfect complex of o_{E_λ}-modules. This implies that $H^1(L_{o_{E_\lambda}})$ is locally free. □

Proof of Theorem 5.10. Define Σ' by $\Sigma' = \Sigma_K \cup \{v\} \cup \{u : u|\ell\}$. First we show 5.10 when the infinity type (k, w) is $((2, \ldots, 2), 0)$.

Let T be the $\mathbb{Q}$-torus $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}$, $U_\infty = T(\mathbb{R})$. For a compact open subgroup U of $T(\mathbb{A}_{\mathbb{Q},f})$, define $S_U(T)$ by

$$S_U(T) = T(\mathbb{Q})\backslash T(\mathbb{A}_{\mathbb{Q}})/U \times U_\infty.$$

For any compact open subgroup $\tilde{K}$ of $G_D(\mathbb{A}_{\mathbb{Q},f})$, the reduced norm $\mathrm{N}_{D/F}$ induces a surjective map

$$\alpha_{\tilde{K}} : S_{\tilde{K}} \xrightarrow{\mathrm{N}_{D/F}} S_{\mathrm{N}_{D/F}(\tilde{K})}(T),$$

which we call the augmentation map.

Since the augmentation maps are functorial with respect to $\tilde{K}$, and $\mathrm{N}_{D/F}(K) = \mathrm{N}_{D/F}(K \cap K_0(v))$, $\alpha_{K\cap K_0(v)} \cdot \mathrm{pr}_i = \alpha_K$ for $i = 1, 2$, which induces a surjective map

$$(*) \qquad \beta : S_K \amalg_{S_{K\cap K_0(v)}} S_K \longrightarrow S_{\mathrm{N}_{D/F}(K)}(T).$$

We show that β is an isomorphism. In the decomposition

$$H^0(S_K, \mathbb{C}) \simeq I^K_{(k,w),\mathbb{C}} \oplus \Big(\bigoplus_{\pi \in \mathscr{A}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))} \pi_f^K \Big),$$

in §4.4, the map

$$H^0(S_K,\ \mathbb{C})^{\oplus 2} \xrightarrow{\mathrm{pr}_1^* + \mathrm{pr}_2^*} H^0(S_{K\cap K_0(v)},\ \mathbb{C})$$

is injective on the part parametrized by $\mathscr{A}_{(k,w)}(G_D(\mathbb{A}_{\mathbb{Q}}))$. By definition, $I^K_{(k,w),\mathbb{C}}$ is identified with $H^0(S_{\mathrm{N}_{D/F}(K)}(T), \mathbb{C})$, and is regarded as the kernel V of $(*)$ via the embedding

$$I^K_{(k,w),\mathbb{C}} \xrightarrow{(i_K, -i_K)} (I^K_{(k,w),\mathbb{C}})^{\oplus 2} \hookrightarrow H^0(S_K,\ \mathbb{C})^{\oplus 2},$$

where $i_K : I^K_{(k,w),\mathbb{C}} \hookrightarrow H^0(S_K,\ \mathbb{C})$ is the inclusion. On the other hand, by Lemma 5.11 (2), V is canonically isomorphic to $H^0(S_{K\cap K_0(v)} \coprod_{S_K} S_{K\cap K_0(v)},\ \mathbb{C})$. So the source and the target of $(*)$ have the same cardinalities, and β must be an isomorphism.

We apply Lemma 5.11 to $\mathrm{pr}_1,\ \mathrm{pr}_2 : S_{K\cap K_0(v)} \to S_K$. It follows that the kernel of

$$H^0(S_K,\ k_\lambda)^{\oplus 2} \xrightarrow{\mathrm{pr}_1^* + \mathrm{pr}_2^*} H^0(S_{K\cap K_0(v)},\ k_\lambda)$$

is canonically isomorphic to $H^0(S_{\mathrm{N}_{D/F}(K)}(T),\ k_\lambda)$. In particular the action of $T_{\Sigma'}$ is of residual type, since the constituents as $D^\times(\mathbb{A}_f^{\Sigma'})$-representations are all one dimensional over $\bar{k}_\lambda$.

We prove the general case. By the definition of $\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}}$, $\bar{\mathscr{F}}_{(k,w),k_\lambda}$ is trivialized in $G_D(\mathbb{A}^\ell_{\mathbb{Q},f})$-equivariant way on $S_{K'}$. Here $K' = K'_\ell \cdot K^\ell$, $K'_\ell = \prod_{u|\ell} K(m_{F_u}) \cap K_u$ if $v \nmid \ell$, $K'_\ell = K_v \cdot \prod_{u|\ell, u\neq v} K(m_{F_u}) \cap K_u$ if $v|\ell$. In the following commutative diagram

$$\begin{array}{ccc} H^0(S_K,\ \bar{\mathscr{F}}^K_{(k,w),k_\lambda})^{\oplus 2} & \xrightarrow{\mathrm{pr}_1^* + \mathrm{pr}_2^*} & H^0(S_{K\cap K_0(v)},\ \bar{\mathscr{F}}^{K\cap K_0(v)}_{(k,w),k_\lambda}) \\ \downarrow & & \downarrow \\ H^0(S_{K'},\ \bar{\mathscr{F}}^{K'}_{(k,w),k_\lambda})^{\oplus 2} & \xrightarrow{\mathrm{pr}_1^* + \mathrm{pr}_2^*} & H^0(S_{K'\cap K_0(v)},\ \bar{\mathscr{F}}^{K'\cap K_0(v)}_{(k,w),k_\lambda}), \end{array}$$

the restriction maps in the vertical arrows are injective. Since the kernel is of residual type in the case of constant coefficient k_λ and $\bar{\mathscr{F}}^{K'}_{(k,w),k_\lambda}$ is trivialized, the general case follows. $\square$

Remark 5.12. *In* [12]*, the universal injectivity (Ihara's Lemma) is proved for division quaternion algebra D over $\mathbb{Q}$ in the following cases: D is definite, with "Eisenstein" instead of "residual type". When D is indefinite, there are some restrictions on ℓ and (k,w) to apply the p-adic Hodge theory.*

The universal injectivity when $q_D = 1$ will be treated, by assuming that there is a finite place $u|\ell$ where D is split.

5.4. **Cohomological universal exactness.** In this paragraph we show a cohomological universal exactness from $K \cap K_{11}(v^{n-1})$ to $K \cap K_{11}(v^n)$ for $n \geq 1$. This case turns out to be easier than the $K_0(v)$-case.

Assume that D is a quaternion algebra over F which defines either a Shimura curve or a Hida variety, and is split at a finite place v. K is an F-factorizable compact open subgroup of $G_D(\mathbb{A}_{\mathbb{Q},f})$, with the v-component $K_v = \mathrm{GL}_2(o_{F_v})'$.

Let $\mathrm{pr}_{i,v} : S_{K_{11}(v^n)\cap K} \to S_{K\cap K_{11}(v^{n-1})}$ $(i = 1,2)$ be two degeneracy maps defined as follows. pr_2 is the canonical projection corresponding to the inclusion $K \cap K_{11}(v^n) \subset K \cap K_{11}(v^{n-1})$, pr_1 is the projection twisted by the conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix}$ at v, that is,
$K_v \cap K_{11}(m_v^n) \to K_v \cap \begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix}^{-1} K_{11}(m_v^n) \begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix} \subset K_v \cap K_{11}(m_{F_v}^{n-1})$.

$\mathrm{pr}'_1,\ \mathrm{pr}'_2 : S_{K\cap K_{11}(v^n)\cap K_0(v^{n+1})} \to S_{K\cap K_{11}(v^n)}$ are both defined in a similar way. We use the same notations in the case of K_1, too.

Proposition 5.13. *Assume that D is a division quaternion algebra over F which satisfies $q_D = \sharp I_D \leq 1$, and is split at a finite place v. K is an F-factorizable compact open small subgroup of $G_D(\mathbb{A}_{\mathbb{Q},f})$, with the v-component $K_v = \mathrm{GL}_2(o_{F_v})'$. For a discrete infinite type (k,w), assume that $k_\iota = 2$ and $w = 0$ for any $\iota \in I_{F,v}$ if $v|\ell$. For $n \geq 1$,*

$$0 \to H^{q_D}(S_{K\cap K_{11}(v^{n-1})},\ \bar{\mathscr{F}}_{(k,w),R}) \xrightarrow{(\mathrm{pr}_2^*, -\mathrm{pr}_1^*)} H^{q_D}(S_{K\cap K_{11}(v^n)},\ \bar{\mathscr{F}}_{(k,w),R})^{\oplus 2}$$
$$\xrightarrow{\mathrm{pr}_1'^* + \mathrm{pr}_2'^*} H^{q_D}(S_{K\cap K_{11}(v^n)\cap K_0(v^{n+1})},\ \bar{\mathscr{F}}_{(k,w),R})$$

is exact for any finite o_{E_λ}-algebra R up to residual type modules. The same is true for K_1.

Proposition 5.13 is proved in [51], lemma 2.5 in the case of $D = M_2(\mathbb{Q})$ and $\mathscr{F}_{(k,w),o_{E_\lambda}} = o_{E_\lambda}$, assuming $v \nmid \ell$.

Proof of Proposition 5.13. We prove the proposition for K_{11}. K_1-case is proved in the same way.

By Proposition 5.2, the cohomology groups commutes with the extension of scalars up to residual type modules. So we may assume that $R = k_\lambda$.

$$\begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix}^{-1} K_{11}(m_{F_v}^{n-1}) \begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix} = \{g \in \mathrm{GL}_2(o_{F_v}); g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a,\ d \equiv 1 \mod m_{F_v}^{n-1},\ b \in m_{F_v},\ c \in m_{F_v}^{n-1}\}$$
$$= K_{11}(m_{F_v}^{n-1}) \cap K^{\mathrm{op}}(m_{F_v})$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix}^{-1} (K_{11}(m_{F_v}^{n}) \cap K_0(m_v^{n+1})) \begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix} = K_{11}(m_{F_v}^{n}) \cap K(m_{F_v})$$

hold. So the commutative diagram

$$(*) \qquad \begin{array}{ccc} S_{K\cap K_{11}(v^n)\cap K_0(v^{n+1})} & \xrightarrow{\mathrm{pr}_1'} & S_{K\cap K_{11}(v^n)} \\ \mathrm{pr}_2' \big\downarrow & & \mathrm{pr}_1 \big\downarrow \\ S_{K\cap K_{11}(v^n)} & \xrightarrow{\mathrm{pr}_2} & S_{K\cap K_{11}(v^{n-1})} \end{array}$$

is factorized into

$$\begin{array}{ccc} S_{K\cap K_{11}(v^n)\cap K_0(v^{n+1})} & \xrightarrow{R(a(p_v)^{-1})} & S_{K\cap K_{11}(v^n)\cap K(v)} \\ \mathrm{pr}_2' \big\downarrow & & \big\downarrow \\ S_{K\cap K_{11}(v^n)} & \xrightarrow{R(a(p_v)^{-1})} & S_{K\cap K_{11}(v^{n-1})\cap K^{\mathrm{op}}(v)} \\ \mathrm{pr}_2 \big\downarrow & & \big\downarrow \\ S_{K\cap K_{11}(v^{n-1})} & \longrightarrow & S_{K\cap K_{11}(v^{n-1})}, \end{array}$$

and $(*)$ is isomorphic to

$$\begin{array}{ccc} Z = S_{K_{11}(v^n)\cap K(v)\cap K} & \xrightarrow{\alpha'} & Y = S_{(K_v\cap K_{11}(m_{F_v}^{n-1})\cap K^0(m_{F_v}))\cdot K^v} \\ \beta' \big\downarrow & & \beta \big\downarrow \\ Y' = S_{K_{11}(v^n)\cap K} & \xrightarrow{\alpha} & X = S_{K_{11}(v^{n-1})\cap K}. \end{array}$$

Here the arrows in the second diagram is obtained by inclusions of compact open subgroups. We write $f = \alpha \cdot \beta' = \beta \cdot \alpha'$.

Lemma 5.14. *In the notations as above,*

(1) *The coproduct $Y \amalg_Z Y'$ is isomorphic to X.*
(2) *For any o_{E_λ}-sheaf $\mathscr{F}$ on X,*
$$0 \longrightarrow \mathscr{F} \longrightarrow \alpha_*\alpha^*\mathscr{F} \oplus \beta_*\beta^*\mathscr{F} \longrightarrow f_*f^*\mathscr{F}$$
is exact.

Proof of Lemma 5.14. We check the claim fiberwise. Take a geometric point x of X. Then the fibers $\alpha^{-1}(x)$, $\beta^{-1}(x)$, and $f^{-1}(x)$ over x are all $K_{11}(m_{F_v}^{n-1}) \cap K_v$-homogeneous spaces, and are identified with the following left $K_{11}(m_{F_v}^{n-1}) \cap K_v$-spaces, respectively: When $n = 1$, $\mathrm{SL}_2(k(v))/A \cdot N$ (A is the group of the scalar matrices represented by a unit of F, N the standard unipotent subgroup consisting of the upper triangular matrices), $\mathrm{SL}_2(k(v))/A \cdot B^{\mathrm{op}}$ (B^{op} is the opposite Borel subgroup consisting of the lower triangular matrices), $\mathrm{SL}_2(k(v))/A$. Since N and B^{op} generates $\mathrm{SL}_2(k(v))$, the claim follows from the

following remark: for a group G and subgroups H_1 and H_2 which generate G, the coproduct $G/H_1 \amalg_G G/H_2$ is a point (the pullback to G of a $\mathbb{C}$-valued function on the coproduct is right H_1 and H_2-invariant, and hence is a constant function). The claim for $n \geq 2$ is proved in the same way.

(2) follows from (1) by applying Lemma 5.11 fiberwise.

$\square$

Consider the complex

$$L = [\alpha_*\alpha^*\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}} \oplus \beta_*\beta^*\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}} \longrightarrow f_*f^*\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}}]$$

where the components are placed in degree -1 and 0. This is a perfect complex of o_{E_λ}-modules on X and the cohomology sheaves are locally constant since all components are locally constant sheaves. By Lemma 5.14, for any o_{E_λ}-sheaf $\mathscr{G}$ on X, $H^{-1}(L \otimes^{\mathbb{L}}_{o_{E_\lambda}} \mathscr{G})$ is isomorphic to $\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}} \otimes_{o_{E_\lambda}} \mathscr{G}$, and hence exact in $\mathscr{G}$. Since non-trivial cohomology sheaves of L are located in degree -1 and 0, $\mathscr{H} = H^0(L)$ is a smooth o_{E_λ}-sheaf. Let $\mathscr{H}'$ be the image of $\alpha_*\alpha^*\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}} \oplus \beta_*\beta^*\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}}$ in $f_*f^*\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}}$.

In summary, we have exact sequences

$$0 \longrightarrow \bar{\mathscr{F}}_{(k,w),o_{E_\lambda}} \longrightarrow \alpha_*\alpha^*\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}} \oplus \beta_*\beta^*\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}} \longrightarrow \mathscr{H}' \longrightarrow 0,$$

and

$$0 \longrightarrow \mathscr{H}' \longrightarrow f_*f^*\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}} \longrightarrow \mathscr{H} \longrightarrow 0.$$

Here, $\mathscr{H}$ and $\mathscr{H}'$ are o_{E_λ}-smooth sheaves, and hence the exactness of the above sequences is preserved by any scalar extensions. In the case of $q_D = 0$, the claim follows immediately.

In the case of $q_D = 1$, we make use of the action of $H_{o_{E_\lambda}} = H(D^\times(\mathbb{A}^\Sigma_{F,f}),\ K^\Sigma)_{o_{E_\lambda}}$, and work in $\bar{\mathscr{C}}_{T_\Sigma}$. As in the proof of Theorem 5.10, we take a finite set of finite places Σ which contains Σ_K, v, and all places dividing ℓ.

In the exact sequence

$$\begin{aligned} H^0(X, \mathscr{H}'_{k_\lambda}) \longrightarrow H^1(X, \bar{\mathscr{F}}_{(k,w),k_\lambda}) &\longrightarrow H^1(X, \alpha_*\alpha^*\bar{\mathscr{F}}_{(k,w),k_\lambda} \oplus \beta_*\beta^*\bar{\mathscr{F}}_{(k,w),k_\lambda}) \\ &\longrightarrow H^1(X, \mathscr{H}'_{k_\lambda}) \longrightarrow H^2(X, \bar{\mathscr{F}}_{(k,w),k_\lambda}) \end{aligned}$$

$H^0(X,\ \mathscr{H}'_{k_\lambda})$ and $H^2(X,\ \bar{\mathscr{F}}_{(k,w),k_\lambda})$ are modules of residual type by Lemma 5.14, and hence vanishes in $\bar{\mathscr{C}}_{T_\Sigma}$.

To show the injectivity of $H^1(X, \mathscr{H}'_{k_\lambda}) \to H^1(X, f_*f^*\bar{\mathscr{F}}_{(k,w),k_\lambda})$, it suffices to prove that $H^0(X, \mathscr{H}_{k_\lambda})$ is of residual type.

$f^*\mathscr{H}_{k_\lambda}$ is an $D^\times(\mathbb{A}^\Sigma_{F,f})$-equivariant quotient of $f^*f_*f^*\bar{\mathscr{F}}_{(k,w),k_\lambda}$, and $f^*f_*f^*\bar{\mathscr{F}}_{(k,w),k_\lambda} = f^*f_*\bar{\mathscr{F}}^{K\cap K_{11}(v^n)\cap K_0(v^{n+1})}_{(k,w),k_\lambda}$ is isomorphic to a direct sum of copies of $\bar{\mathscr{F}}^{K\cap K_{11}(v^n)\cap K_0(v^{n+1})}_{(k,w),k_\lambda}$ in a $D^\times(\mathbb{A}^\Sigma_{F,f})$-equivariant way. Thus $H^0(Z, f^*\mathscr{H}_{k_\lambda})$ is of residual type by Lemma 5.14. $H^0(X, \mathscr{H}_{k_\lambda})$ is a submodule of $H^0(Z, f^*\mathscr{H}_{k_\lambda})$ and hence vanishes in $\bar{\mathscr{C}}_{T_\Sigma}$.

$\square$

6. Nearly ordinary automorphic representations

6.1. **Renormalization of cohomological Hecke correspondences.** We fix a place v which divides ℓ. For $K = \mathrm{GL}_2(o_{F_v})$, we have the Cartan decomposition

$$\mathrm{GL}_2(F_v) = \cup_{(a,b)\in\mathbb{Z}^2, a\leq b} K \begin{pmatrix} p_v^a & 0 \\ 0 & p_v^b \end{pmatrix} K.$$

Definition 6.1. (1) *For an infinity type (k,w) and a place v dividing ℓ, the v-type is defined as a pair (k_v, w), where $k_v = (k_\iota)_{\iota \in I_{F,v}}$. Here we regard $I_{F,v}$ (cf. §4.3) as a subset of $I_{F,\infty}$ by $\bar{E}_\lambda \simeq \mathbb{C}$.*

(2) *For $h \in \mathrm{GL}_2(F_v)$, a pair of integers $(a_v(h), b_v(h))$ is defined by $KhK = K \begin{pmatrix} p^{a_v(h)} & 0 \\ 0 & p_v^{b_v(h)} \end{pmatrix} K$ for $K = \mathrm{GL}_2(o_{F_v})$. $(a_v(h), b_v(h))$ depends only on h under the condition that $a_v(h) \leq b_v(h)$.*

(3) *For an element $g \in G_D(\mathbb{A}_{\mathbb{Q},f})$,*

$$m_v(g) = \prod_{\iota \in I_{F,v}} (\iota(p_v))^{w \cdot a_v(g_v) + k'_\iota \cdot (b_v(g_v) - a_v(g_v))}$$

is the multiplier of g at v with respect to the v-type (k_v, w) and p_v. Here k'_ι is defined as 4.1, and $k'_v = (k'_\iota)_{\iota \in I_{F,v}}$.

Then $(g_v)^{-1}(V_{(k,w),o_{E_\lambda}}) \subset m_v(g) \cdot V_{(k,w),o_{E_\lambda}}$, and $(R(g^{-1}))^* \bar{\mathscr{F}}_{(k,w),o_{E_\lambda}} \subset \prod_{v|\ell} m_v(g_v) \bar{\mathscr{F}}_{(k,w),o_{E_\lambda}}$. For an F-factorizable compact open subgroup K of $G_D(\mathbb{A}_{\mathbb{Q},f})$, one defines a cohomological correspondence of $\bar{\mathscr{F}}_{(k,w),o_{E_\lambda}}$ by using $(\prod_{v|\ell} m_v(g_v))^{-1} R(g^{-1})^*$ at $v|\ell$ when K_v is a subgroup of $\mathrm{GL}_2(o_{F_v})$.

Definition 6.2. (1) *For an F-factorizable compact open subgroup K of $G_D(\mathbb{A}_{\mathbb{Q},f})$, the renormalized cohomological correspondence $[KgK]^{\mathrm{ren}}$ is the correspondence given by $(\prod_{v|\ell} m_v(g_v))^{-1} R(g^{-1})^*$.*

(2) *For a uniformizer p_v of F_v, $\tilde{U}(p_v)$ and $\tilde{U}(p_v, p_v)$ is the renormalized $U(p_v)$ and $U(p_v, p_v)$-operators defined by $[Ka(p_v)K]^{\mathrm{ren}}$ and $[Kb(p_v)K]^{\mathrm{ren}}$, respectively.*

Note that this renormalization depends on the choice of infinity type (k,w), and the choice of an isomorphism $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$.

Since the renormalized correspondence preserves the o_{E_λ}-lattices, we have

Proposition 6.3. *Let π be a representation in $\mathscr{A}_{(k,w)}(G_D)$ with a discrete infinity type. Assume that D is split at the places dividing ℓ. For any F-factorizable compact open subgroup K, and any eigenvalue α of $\chi_{KgK} : \pi^K \to \pi^K$, the inequality*

$$\mathrm{val}(\alpha) \geq \sum_{v|\ell} \mathrm{val}(m_v(g))$$

$$= w \sum_{v|\ell} [F_v : \mathbb{Q}_\ell] \cdot a_v(g_v) \mathrm{val}(p_v) + \sum_{v|\ell} (\sum_{\iota \in I_{F,v}} k'_\iota \cdot (b_v(g_v) - a_v(g_v))) \mathrm{val}(p_v)$$

holds. Here $\mathrm{val} : \bar{E}_\lambda^\times \to \mathbb{Q}$ *is the additive valuation.*

At v, we normalize val by $\mathrm{val}(\iota(p_v)) = 1$ for $\iota \in I_{F,v}$. This definition does not depend on the choice of a uniformizer.

For a quasi-character $\mu : F_v^\times \to \bar{E}_\lambda^\times$, the slope of μ is defined by

$$\mathrm{slope}(\mu) = \mathrm{val}(\mu(p_v)),$$

where p_v is a uniformizer at v.

Corollary 6.4 (Hida, [23], theorem 4.11)**.** *Let π be a representation in $\mathscr{A}_{(k,w)}(G_D)$ with a discrete infinity type, ρ_{F_v} the F-semi-simple representation of the Weil-Deligne group W'_F over $\bar{E}_\lambda$ corresponding to π_v by the local Langlands correspondence* [29]*. Assume that χ is a subrepresentation of ρ_{F_v}. Then the inequality*

$$\mathrm{slope}(\chi) \geq \sum_{\iota \in I_{F,v}} k'_\iota$$

holds.

Hida proves the theorem including limit discrete series by using Hilbert modular varieties, that is, in the case when $k_\iota \geq 1$ for all $\iota \in I_{F,\infty}$.

6.2. **Nearly ordinary automorphic representations.** We consider the local restrictions of the Galois representation attached to automorphic representations when the residual characteristic p is ℓ, and define the notion of minimality. In this article, nearly ordinary representations are mainly considered for $v|\ell$.

Let π be a cuspidal representation $\mathrm{GL}_2(\mathbb{A}_F)$ of discrete infinity type (k,w). The local restriction of the associated ℓ-adic representations $\rho_{\pi,\lambda}|_{G_{F_v}}$ for $v|\ell$ is a potentially stable representation in the sense of Fontaine unless $k=(2,\dots 2)$ (see [42]. It is enough to show this over some finite extension of F_v, and the claim follows from Carayol [6], Blasius-Rogawski [3], and de Jong's method of alteration [8] and Tsuji's theorem [48]). If $k=(2,\dots,2)$, we assume that d is odd, or π admits an essentially square integrable local component. For potentially stable ℓ-adic representations, T. Saito [41] has constructed a representation of the Weil-Deligne group W'_{F_v} over $\bar{E}_\lambda$.

Here we make a temporary construction in some one dimensional cases. We assume that all continuous field embeddings $F_v \hookrightarrow \bar{E}_\lambda$ factors through E_λ.

For a quasi-character $\mu : F_v^\times \to E_\lambda^\times$ and an integral vector $k'_v = (k'_\iota)_{\iota\in I_{F,v}} \in \mathbb{Z}^{I_{F,v}}$ which satisfies

$$\mathrm{slope}(\mu) = \sum_{\iota\in I_{F,v}} k'_\iota,$$

we define the associated ℓ-adic character $L_{k'_v}(\mu)$.

Let $\mathscr{L}(p_v)$ be the Lubin-Tate formal group over o_{F_v} attached to p_v, $\chi_{p_v} : G_{F_v} \to o_{F_v}^\times \hookrightarrow o_{E_\lambda}^\times$ the Galois representation attached to p_v-divisible group $\mathscr{L}(p_v)[p_v^\infty]$. It is explicitly given as follows. The splitting $\mathbb{Z} \to F_v^\times$ of

$$1 \longrightarrow o_{F_v}^\times \longrightarrow F_v^\times \xrightarrow{\mathrm{val}} \mathbb{Z} \longrightarrow 0$$

given by $1 \mapsto p_v$ induces the projection $\alpha : \widehat{F_v^\times} \twoheadrightarrow o_{F_v}^\times$. $\alpha(x) = x\cdot p_v^{-\mathrm{val}(x)}$ for $x \in F_v^\times$.

By our normalization of the Galois module associated to p-divisible groups, χ_{p_v} is the composition of $G_{F_v}^{\mathrm{ab}} \simeq \widehat{F_v^\times} \xrightarrow{\alpha} o_{F_v}^\times \xrightarrow{(-)^{-1}} o_{F_v}^\times \hookrightarrow o_{E_\lambda}^\times$.

We define a quasi-character $\mu_{p_v} : F_v^\times \to E_\lambda^\times$ by

$$\mu_{p_v}(x) = \mu(x)/\prod_{\iota\in I_{F,v}} \iota(p_v^{\mathrm{val}(x)})^{k'_\iota}.$$

Then μ_{p_v} takes the values in $o_{E_\lambda}^\times$, extends to a continuous character of $\widehat{F_v^\times}$, and is regarded as a character of G_{F_v} by the local class field theory.

$$L_{k'_v}(\mu) = \mu_{p_v}\cdot \prod_{\iota\in I_{F,v}} (\iota\circ\chi_{p_v})^{k'_\iota} : G_{F_v} \longrightarrow o_{E_\lambda}^\times$$

is independent of any choice of p_v.

Definition 6.5 (Hida). *Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of infinity type (k,w) defined over $\bar{E}_\lambda$, and v a finite place of F which divides ℓ.*

(1) *Let ρ_v be the F-semi-simple representation of the Weil-Deligne group W'_{F_v} over $\bar{E}_\lambda$ which corresponds to π_v by the local Langlands correspondence. We say π is nearly ordinary at v if ρ_v contains a character μ_v as a subrepresentation, and the slope of μ_v satisfies*

$$\mathrm{slope}(\mu_v) = \sum_{\iota \in I_{F,v}} k'_\iota.$$

The character μ_v is called a nearly ordinary character of π at v.

(2) *Assume that π is nearly ordinary at v with nearly ordinary character μ_v. A non-zero vector z in the representation space V of π_v is called a nearly ordinary vector if*

$$\pi_v(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix}) \cdot z = \chi_{1,v}(a)|a|_v \chi_{2,v}(d) z, \qquad a,\ d \in F_v^\times,$$

holds for some quasi-character $\chi_{1,v}$, and $\chi_{2,v} = \mu_v$. $\mu_v|_{o_{F_v}^\times}$ is a nearly ordinary type of π at v.

(3) *For a quaternion algebra D central over F, assume that D is split at the places dividing ℓ. Then for a place v dividing ℓ and a representation $\pi \in \mathscr{A}_{(k,w)}(G_D)$, π is nearly ordinary at v if the Jacquet-Langlands correspondent* $\mathrm{JL}(\pi)$ *is nearly ordinary at v.*

In other words, π is nearly ordinary at v if and only if $\tilde{U}(p_v)$-operator has an ℓ-adic unit eigenvalue. In the case when $k_\iota \geq 2$ for any $\iota \in I_{F,v}$, a nearly ordinary character μ_v is unique, since $\mathrm{slope}(\chi_{1,v}) = \mathrm{slope}(\chi_{2,v}) + \sum_{\iota \in I_{F,v}} (k_\iota - 1)$ holds.

Conjecture 6.6. *For a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of infinity type (k, w), assume that π is nearly ordinary at v with nearly ordinary character μ_v. Let $\rho = \rho_{\pi,E_\lambda}$ be the ℓ-adic representation associated to π. Then the local representation $\rho|_{F_v}$ is reducible, and contains $L_{k'_v}(\mu_v)$ as a subrepresentation.*

Hypothesis 6.7 (Local monodromy hypothesis)**.** *Conjecture 6.6 is true for π with discrete infinity type (k, w) if one of the following conditions hold:*

- *The degree d is odd.*
- *d is even, and π has an essentially square integrable component π_u for some finite place u.*

Hypothesis 6.7 is known to hold in many cases.

Theorem 6.8. *Hypothesis 6.7 is true in the following cases:*

(1) *π is nearly ordinary at all $v|\ell$.*

(2) *The infinity type of π is $((2, \ldots, 2), 0)$, and the degree d is odd, or d is even and π has an essentially square integrable component at some finite place u (*[51]*, lemma 2.1.5).*

The case of (1) follows from the case of (2) by Hida's control theorem of nearly ordinary Hecke algebras [24], and the existence of Galois representations. The method of [51] is used there. Since π is obtained as a specialization of Hida family, it suffices to check Hypothesis 6.7 for the Galois representation attached to the nearly ordinary Hecke algebra. Since the algebraic points corresponding to cuspidal representations with infinity type $((2, \ldots, 2), 0)$ are dense, the claim follows from (2). For (2), see [51], theorem 2.1.4 when π_v belongs to principal series. When π_v is a (twisted) special representation, (2) follows from [21].

Remark 6.9. *A proof of Hypothesis 6.7 will be given if the residual representation is not of residual type in a forthcoming article. For $v|\ell$, we construct an ℓ-adic family of cuspidal representations which are nearly ordinary at v, and reduce to the case of (2).*

6.3. **Modularity and minimality.** As in the introduction, an absolutely irreducible mod ℓ Galois representation $\bar{\rho} : G_\Sigma \to \mathrm{GL}_2(k_\lambda)$ is modular if there is an ℓ-adic field E_λ and a cuspidal representation π on $\mathrm{GL}_2(\mathbb{A}_F)$ of discrete infinity type (k, w) such that the finite part π_f is defined over E_λ, and $\bar{\rho} \simeq \rho_{\pi,E_\lambda} \mod \lambda$ over $\bar{k}_\lambda$. In §7, we need to choose π with a distingushed property. We make the following temporaly definition for such a choice.

For $s \in \mathbb{C}$, $\omega_s : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}$ is the Hecke character defined by

$$\omega_s(x) = |x|^s.$$

If s is an integer, this is an algebraic Hecke character, and the ℓ-adic representation associated to ω_s is χ_{cycle}^s by our normalization of the reciprocity map.

Definition 6.10. *Let α be an integer.*

(1) *For a character $\bar{\mu} : G_F \to k_\lambda^\times$, let $(\bar{\mu})_{\mathrm{lift}}$ be the Teichmüller lift of $\bar{\mu}$. We view $\nu = (\bar{\mu})_{\mathrm{lift}} \cdot (\chi_{\mathrm{cycle}}^\ell)^{w+1}$ as a Hecke character of finite order by the class field theory. Then the algebraic Hecke character associated to $\bar{\mu}$ of weight 2α is defined as*

$$\chi_{\bar{\mu},\alpha}^{\mathrm{Hecke}} = \nu \cdot \omega_{-\alpha}.$$

(2) *For $\bar{\rho} : G_F \to \mathrm{GL}_2(k_\lambda)$, and an integer α, $\chi_{\det\bar{\rho},\alpha}^{\mathrm{Hecke}}$ is denoted by $\chi_{\bar{\rho},\alpha}^{\mathrm{Hecke}}$.*

By the definition, the λ-adic representation $\chi_{\bar{\mu},\alpha} : G_F \to o_{E_\lambda}^\times$ of weight 2α attached to $\chi_{\bar{\mu},\alpha}^{\mathrm{Hecke}}$ is given by

$$\chi_{\bar{\rho},\alpha} = (\det\bar{\rho})_{\mathrm{lift}} \cdot (\chi_{\mathrm{cycle},\ell})^{-\alpha}.$$

Definition 6.11. *Let $\bar{\rho}$ be a modular mod ℓ-representation with a minimal deformation type $\mathscr{D}_{\min}$. A cuspidal representation $\pi_{\min}$ of $\mathrm{GL}_2(\mathbb{A}_F)$ with a discrete infinity type (k, w) defined over E_λ is called a minimal lift of $\bar{\rho}$ if the following properties hold:*

(1) *ρ_{π,E_λ} is a deformation of $\bar{\rho}$.*
(2) *$\rho_{\pi,E_\lambda}|_{G_{F_v}}$ for $v \nmid \ell$ is a finite deformation of $\bar{\rho}|_{F_v}$.*
(3) *The central character of $\pi_{\min}$ is $\chi_{\bar{\rho},w}^{\mathrm{Hecke}}$.*
(4) *If $\mathrm{def}_{\mathscr{D}_{\min}}(v) = \mathbf{fl}$, the v-type (k_v, w) is $((2,\dots,2), w)$, and $\pi_v \otimes \mu_v^{-1}$ is spherical for some quasi-character μ_v. $\mu_v \cdot |\cdot|_v^{\frac{w}{2}}$ takes the values in $o_{E_\lambda}^\times$, $\mu_v|_{o_{F_v}^\times}$ has order prime to ℓ. μ_v is called a flat twist character, and $\mu_v|_{o_{F_v}^\times}$ the flat twist type of π at v.*
(5) *If $v|\ell$ and $\mathrm{def}_{\mathscr{D}_{\min}}(v) = \mathbf{n.o.f.}, \mathbf{n.o.}$, $\pi_{\min}$ is nearly ordinary at v. For the nearly ordinary character μ_v, the nearly ordinary type $\mu_v|_{o_{F_v}^\times}$ has the order prime to ℓ. $L_{k'_v}(\mu_v)$ is a lift of the nearly ordinary type $\bar{\kappa}_v$ of $\bar{\rho}|_{F_v}$.*
(6) *If $\mathrm{def}_{\mathscr{D}_{\min}}(v) = \mathbf{n.o.f.}$, the v-type (k_v, w) is $((2,\dots,2), w)$, and $\pi_v \otimes \mu_v^{-1}$ is spherical for the nearly ordinary character μ_v.*

Remark 6.12. (1) *Assume the following condition is satisfied at $v \nmid \ell$: $\bar{\rho}|_{G_{F_v}}$ is of type 0_{NE}, or $\bar{\rho}|_{G_{F_v}}$ is absolutely reducible and ramified at v with $\dim_{k_\lambda} \bar{\rho}^{I_{F_v}} = 1$.*
Then the condition (2) for $v \nmid \ell$ is equivalent to the equality of conductors

$$\mathrm{cond}\pi_v = \mathrm{Art}\bar{\rho}|_{I_{F_v}}.$$

(2) *If the conditions (4)-(6) at $v|\ell$ is satisfied for a modular lift π, by the results of* [15], [25], [26], [35], *one finds $\pi_{\min}$ so that (2) and (3) are satisfied.*

We define the K-type $(G_{\bar{\rho}|_{F_v}}, K_*(\bar{\rho}|_{F_v}), \nu_*(\bar{\rho}|_{F_v}))$ for $* = \mathbf{fl}, \mathbf{n.o.f.}, \mathbf{n.o.}$ from $\bar{\rho} : G_F \to \mathrm{GL}_2(k_\lambda)$ and the v-type (k_v, w).

Definition 6.13. *Let $\bar{\rho}$ be a mod ℓ-representation with a deformation type $\mathscr{D}$ and a v-type (k_v, w).*

(1) *When $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{fl}$, $k_v = (2, \dots, 2)$, and for the flat twist character $\bar{\kappa}_v$ of $\bar{\rho}|_{F_v}$, a quasi-character χ is called an automorphic flat twist character if $\chi \cdot | \cdot |_v^{\frac{w}{2}}$ takes the values in o_{E_λ} and the reduction mod λ is $\bar{\kappa}_v \cdot \bar{\chi}_{\mathrm{cycle}}^{\frac{w}{2}}$. $\nu_v = \chi|_{o_{F_v}^\times}$ is called the automorphic flat twist type at v.*

(2) *When $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{n.o.f.}, \mathbf{n.o.}$, for the nearly ordinary character $\bar{\kappa}_v$ of $\bar{\rho}|_{F_v}$, a quasi-character χ of slope $\sum_{\iota \in I_{F,v}} k'_\iota$ is called an automorphic nearly ordinary character if $\chi|_{o_{F_v}^\times}$ has order prime to ℓ, and $L_{k'_v}(\chi) \mod \lambda = \bar{\kappa}_v$. $\nu_v = \chi|_{o_{F_v}^\times}$ is called the automorphic nearly ordinary type at v.*

An automorphic twist type is unquely determined from $\bar{\rho}$ and (k_v, w).

Definition 6.14. *Let $\bar{\rho}$ be a mod ℓ-representation with a deformation type $\mathscr{D}$ and a discrete infinity type (k, w).*

(1) *$G_{\bar{\rho}|_{F_v}} = \mathrm{GL}_2(F_v)$ for $v|\ell$.*

(2) *If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{fl}$, let ν_v be the automorphic flat twist type of $\bar{\rho}$. Then*
$$K_{\mathbf{fl}}(\bar{\rho}|_{F_v}) = \mathrm{GL}_2(o_{F_v})$$
with the K-character $\nu_{\mathbf{fl}}(\bar{\rho}|_{F_v}) : K_{\mathbf{fl}}(\bar{\rho}|_{F_v}) \xrightarrow{\det} o_{F_v}^\times \xrightarrow{\nu_v} o_{\mathscr{D}}^\times$.

(3) *If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{n.o.f.}$ let ν_v be the automorphic nearly ordinary type of $\bar{\rho}$. Then*
$$K_{\mathbf{n.o.f.}}(\bar{\rho}|_{F_v}) = \mathrm{GL}_2(o_{F_v})$$
with the K-character $\nu_{\mathbf{n.o.f.}}(\bar{\rho}|_{F_v}) : K_{\mathbf{n.o.f.}}(\bar{\rho}|_{F_v}) \xrightarrow{\det} o_{F_v}^\times \xrightarrow{\nu_v} o_{\mathscr{D}}^\times$.

(4) *If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{n.o.}$ let ν_v be the automorphic nearly ordinary type of $\bar{\rho}$. Then*
$$K_{\mathbf{n.o.}}(\bar{\rho}|_{F_v}) = \mu_{\ell^\infty}(F) \cdot K_1(m_{F_v})$$
with the K-character $\nu_{\mathbf{n.o.}}(\bar{\rho}|_{F_v}) : K_{\mathbf{n.o.}}(\bar{\rho}|_{F_v}) \xrightarrow{\det} o_{F_v}^\times \xrightarrow{\nu_v} o_{\mathscr{D}}^\times$.

Similarly as in §3.7, the intertwining space is defined as
$$I_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v) = \mathrm{Hom}_{K_{\mathrm{def}_{\mathscr{D}}(v)}}(\nu_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}), \pi_v).$$

As an analogue of proposition 3.29, we have

Proposition 6.15. *For a cuspidal representation π of infinity type (k, w) giving $\bar{\rho}$, assume that π is nearly ordinary at v and we are in case 6.11, (4). Then the part $\tilde{U}(p_v)$-operator has an ℓ-adic unit eigenvalue in $I_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v)$ is at most one dimensional.*

Proof of Proposition 6.15. By twisting, we may assume that the twist type at v is trivial. The only non-trivial statement in 6.15 is the case when π_v is spherical and $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{n.o.}$, since $I_{\mathbf{n.o.}}(\bar{\rho}|_{F_v}, \pi_v)$ is two dimensional. For the two eigenvalues α_v, β_v, slopes are different as in §6.2. So the only one eigenvalue of $\tilde{U}(p_v)$-operator is an ℓ-adic unit. □

Remark 6.16. *If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{n.o.}, \mathbf{n.o.f.}$ and $\bar{\rho}|_{F_v}$ is semi-simple, we must specify the nearly ordinary type as in §3.8, since the possibility is not unique. So the notion of minimal lift depends on this choice.*

7. Universal modular deformations

As is recalled in the introduction, the existence of the λ-adic representation $G_F \to \mathrm{GL}_2(T \otimes_{o_{E_\lambda}} E_\lambda)$ is known, where T is a certain ℓ-adic Hecke algebra attached to automorphic representations. It is quite important to know the existence of a Galois representation having the values in $\mathrm{GL}_2(T)$, which can be seen as a deformation of a mod ℓ-representation.

In this section, we construct it when the residual representation is absolutely irreducible, and study the local deformation properties. The local behavior is controled by the compatibility of the local and the global Langlands correspondences at the places outside ℓ. At a place dividing ℓ, we need much finer information beyond the general theory still.

The discovery of Galois representations having values in ℓ-adic Hecke algebras is due to Hida (in the case of $\mathbb{Q}$, and for ordinary Hecke algebras [22]). We use the method of pseudo-representations of Wiles [51] to construct the Galois representation.

7.1. **Hecke algebras.** Let $\mathscr{S}_{(k,w)}(G_D)$ be a $G_D(\mathbb{A}_{\mathbb{Q},f})$-representation over $\bar{E}_\lambda$ defined by

$$\mathscr{S}_{(k,w)}(G_D) = \bigoplus_{\pi \in \mathscr{A}_{(k,w)}(G_D)} \pi_f.$$

Here $\mathscr{S}_{(k,w)}(G_D)$ is defined up to isomorphisms, and for a compact open subgroup K of $G_D(\mathbb{A}_{\mathbb{Q},f})$, the K-fixed part $\mathscr{S}_{(k,w)}(G_D)^K$ is finite dimensional over $\bar{E}_\lambda$.

For an algebraic Hecke character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$, $\mathscr{S}_{(k,w),\chi}(G_D)$ is defined as the subspace parametrized by $\mathscr{A}_{(k,w),\chi}(G_D)$.

For an F-factorizable compact open subgroup K of $G_D(\mathbb{A}_{\mathbb{Q},f})$ and a finite set of finite places Σ which contains Σ_K, the reduced Hecke algebra generated by the standard Hecke operators is defined by

$$T^{\ss}_{K,\Sigma} = \mathbb{Z}[T_v,\ T_{v,v},\ v \notin \Sigma] \subset \mathrm{End}_{\mathbb{Z}} \mathscr{S}_{(k,w)}(G_D)^K,$$

and for a commutative ring R,

$$T^{\ss}_{K,\Sigma,R} = T^{\ss}_{K,\Sigma} \otimes_{\mathbb{Z}} R.$$

$T^{\ss}_{K,\Sigma}$ is commutative by the definition, and $T^{\ss}_{K,\Sigma,\bar{E}_\lambda}$ acts faithfully on $\mathscr{S}_{(k,w)}(G_D)^K$. It is known that $T^{\ss}_{K,\Sigma}$ is finite free over $\mathbb{Z}$ since it fixes a $\mathbb{Z}$-lattice in $\mathscr{S}_{(k,w)}(G_D)^K$.

For an ℓ-adic integer ring o_λ and Σ which contains all places dividing ℓ, an o_λ-algebra homomorphism $f : T^{\ss}_{K,\Sigma,o_\lambda} \to o_\lambda$ corresponds to an admissible irreducible $D^\times(\mathbb{A}^\Sigma_{F,f})$-representation of the form $\pi^\Sigma = \bigotimes_{v \notin \Sigma} \pi_v$ for some cuspidal representation $\pi \in \mathscr{A}_{(k,w)}(G_D)$ defined over E_λ. By the strong multiplicity one theorem for GL_2 and their inner twists, π is uniquely determined from f, and f is denoted by f_π. In this way we may identify such an o_λ-homomorphism f with a representation $\pi \in \mathscr{A}_{(k,w)}(G_D)$ which is defined over E_λ and satisfies $(\pi_f)^K \neq \{0\}$.

In terms of the Hecke algebra homomorphism f_π attached to $\pi \in \mathscr{A}_{(k,w)}(G_D)$, the Galois representation $\rho_{\pi,\bar{E}_\lambda}$ is described as follows. For an F-factorizable compact open subgroup K such that $\pi_f^K \neq \{0\}$, and a finite set of finite places Σ which contains $\Sigma_K \cup \{v : v|\ell\}$,

$$\begin{cases} \mathrm{trace}\rho_{\pi,\bar{E}_\lambda}(\mathrm{Fr}_v) = f_\pi(T_v) \\ \det\rho_{\pi,\bar{E}_\lambda}(\mathrm{Fr}_v) = q_v \cdot f_\pi(T_{v,v}) \quad \text{for any } v \notin \Sigma \end{cases}$$

holds. The field of definition E_π is generated by $f_\pi(T_v)$ and $f_\pi(T_{v,v})$ for $v \notin \Sigma$, and the embedding $E_\pi \hookrightarrow \bar{E}_\lambda$ determines a valuation $\tilde{\lambda}$ of E_π. The Galois representation $\rho_{\pi,\bar{E}_\lambda}$ is defined over the completion $E_{\pi,\lambda'}$ with respect to λ'.

7.2. **ℓ-adic Hecke algebras attached to deformation types.** Let F be a totally real field, E_λ an ℓ-adic field with $\ell \geq 3$, $\bar{\rho} : G_F \to \mathrm{GL}_2(k_\lambda)$ an absolutely irreducible representation. Let $\mathscr{D}$ be a deformation type of $\bar{\rho}$. In the following sections, we fix a discrete infinity type (k,w) which satisfies

IT If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{fl}$ (resp. **n.o.f.**) for $v|\ell$, the v-type $(k_v, w) = ((2,\ldots,2), w)$.

We restrict ourselves to deformation types which satisfy the following conditions:

D1 $o_{\mathscr{D}}$ is an ℓ-adic integer ring.

D2 If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{fl}$ for $v|\ell$, the flat twist type $\kappa_{\mathscr{D},v}$ is $L_{k'_v}(\chi_v)|_{I_{F_v}}$. Here χ_v is an automorphic flat twist character (see Definition 6.13).

D3 If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{n.o.f}, \mathbf{n.o.}$ for $v|\ell$, the nearly ordinary type $\kappa_{\mathscr{D},v}$ is $L_{k'_v}(\chi_v)|_{I_{F_v}}$. Here χ_v is an automorphic nearly ordinary character (see Definition 6.13).

We view the quotient field $E_{\mathscr{D}}$ as a subfield of $\bar{E}_\lambda$.

Moreover, we assume the following conditions on $\bar{\rho}$:

MM1 There is a minimal deformation type $\mathscr{D}_{\min}$ such that $o_{\mathscr{D}_{\min}} = o_{E_\lambda}$.

MM2 There is a cuspidal representation $\pi_{\min}$ of $\mathrm{GL}_2(\mathbb{A}_F)$ with a discrete infinity type (k,w) such that $\pi_{\min,f}$ is defined over E_λ, and $\pi_{\min}$ is a minimal lift of $\bar{\rho}$ (see Definition 6.11).

Note that there is a morphism $\mathscr{D}_{\min} \to \mathscr{D}$ in $\mathrm{Type}(\bar{\rho})$ (Definition 3.32) for some minimal deformation type $\mathscr{D}_{\min}$.

We attach a triple $(G_{\bar{\rho}}, K_{\mathscr{D}}, \nu_{\mathscr{D}})$ to $\mathscr{D}$, where $G_{\bar{\rho}}$ is an inner twist of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}$ which depends only on $\bar{\rho}$, $K_{\mathscr{D}}$ is a compact open subgroup of $G_{\bar{\rho}}(\mathbb{A}_{\mathbb{Q},f})$, and $\nu_{\mathscr{D}} : K_{\mathscr{D}} \to o_{\mathscr{D}}^\times$ is a continuous character.

We choose a division quaternion algebra D which satisfies the following conditions:

- For a finite place v, D_{F_v} is split unless $\bar{\rho}|_{F_v}$ is of type 0_E.
- If $\bar{\rho}|_{F_v}$ is of type 0_E at a finite place v, D_{F_v} is a quaternion algebra over F_v with the invariant $\frac{c}{2} \mod \mathbb{Z}$, where c is the relative conductor of $\bar{\rho}|_{F_v}$.
- $q_D = \sharp I_D \leq 1$.

A quaternion algebra D which satisifies the above conditions exists. We make a choice from such quaternion algebras, and denote it by $D(\bar{\rho})$.

$$q_{D(\bar{\rho})} \equiv \sharp\{v : \ \bar{\rho}|_{F_v} \text{ is of type } 0_E, \text{ and the relative conductor is odd}\} + d \mod 2$$

holds for $d = [F:\mathbb{Q}]$, and $D(\bar{\rho})$ is split at all places dividing ℓ.

$G_{\bar{\rho}}$ is defined as $\mathrm{Res}_{F/\mathbb{Q}} D(\bar{\rho})^\times$. $K_{\mathscr{D}}$ is defined as the product $\prod_{v \in |F|_f} K_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v})$, and $\nu_{\mathscr{D}} : K_{\mathscr{D}} \to o_{\mathscr{D}}^\times$ is the product of $\nu_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}) : K_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}) \to o_{\mathscr{D}}^\times$. Here $(K_*(\bar{\rho}|_{F_v}), \nu_*(\bar{\rho}|_{F_v}))$ is a pair of the compact open subgroup of $D(\bar{\rho})^\times(F_v)$ and the K-character defined in §3.7 for $* = \mathbf{f}, \mathbf{u}$, and in §6.3 for $* = \mathbf{n.o.f.}, \mathbf{n.o.}, \mathbf{fl}$.

Definition 7.1. *For an absolutely irredicible representation $\bar{\rho}$ which satisfies* **MM1** *and* **MM2**, *let $(G_{\bar{\rho}}, K_{\mathscr{D}}, \nu_{\mathscr{D}})$ be a triple defined as above.*

(1) *$I_{(k,w),\mathscr{D}}$ is the intertwining space defined by*

$$I_{(k,w),\mathscr{D}} = \mathrm{Hom}_{K_{\mathscr{D}}}(\nu_{\mathscr{D}}, \mathscr{S}_{(k,w)}(G_{\bar{\rho}})).$$

(2) *For $\pi \in \mathscr{A}_{(k,w)}(G_{\bar{\rho}})$, $I_{(k,w),\mathscr{D}}(\pi) = \mathrm{Hom}_{K_{\mathscr{D}}}(\nu_{\mathscr{D}}, \pi_f)$.*

(3) *For an algebraic Hecke character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \bar{E}_\lambda^\times$, $I_{(k,w),\mathscr{D},\chi}$ is the subspace which corresponds to $\mathscr{S}_{(k,w),\chi}(G_{\bar{\rho}})$.*

$I_{(k,w),\mathscr{D}}$ is isomorphic to $\oplus_{\pi \in \mathscr{A}_{(k,w)}(G_{\bar{\rho}})} I_{(k,w),\mathscr{D}}(\pi)$, and each $I_{(k,w),\mathscr{D}}(\pi)$ is isomorphic to the tensor product of the local intertwining spaces $\otimes_{v \in |F|_f} I_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v)$.

Definition 7.2. *Let $\bar{\rho}$ be an absolutely irreducible representation which satisfies* **MM1** *and* **MM2**.

(1) *We define a decomposition of $\Sigma_{\mathscr{D}} = P^{\mathbf{fl}} \cup P^{\mathbf{n.o.}} \cup P^{\mathbf{f}}_{\mathscr{D}} \cup P^{\mathbf{u}}_{\mathscr{D}}$ by $P^{\mathbf{fl}} = \mathrm{def}_{\mathscr{D}}^{-1}(\mathbf{fl})$, $P^{\mathbf{n.o.}} = \mathrm{def}_{\mathscr{D}}^{-1}(\{\mathbf{n.o.}, \mathbf{n.o.f.}\})$, $P^{\mathbf{f}}_{\mathscr{D}} = \mathrm{def}_{\mathscr{D}}^{-1}(\mathbf{f})$, and $P^{\mathbf{u}}_{\mathscr{D}} = \mathrm{def}_{\mathscr{D}}^{-1}(\mathbf{u})$.*

$P^{\mathrm{exc}} = \{\bar{\rho}|_{F_v}$ is of type 0_E, and the relative conductor of $\bar{\rho}$ is odd.$\}$ is a subset of $P^{\mathbf{f}}_{\mathscr{D}} \cup P^{\mathbf{u}}_{\mathscr{D}}$.

(2) *The Hecke algebra* $\tilde{T}_{\mathscr{D}}$ *is the* $o_{\mathscr{D}}$*-algebra generated by the following elements in* $\mathrm{End}_{o_{\mathscr{D}}} I_{(k,w),\mathscr{D}}$*:* T_v *and* $T_{v,v}$ *for* $v \notin \Sigma_{\mathscr{D}}$*,* $U(p_v)$ *and* $U(p_v, p_v)$ *for* $v \in P^{\mathbf{u}}_{\mathscr{D}} \setminus P^{\mathrm{exc}}$*, and* $\tilde{U}(p_v)$ *and* $\tilde{U}(p_v, p_v)$ *for* $v \in P^{\mathbf{n.o.}}$*.*

The Hecke algebra $\tilde{T}_{\mathscr{D},\chi}$ *with a fixed central character* χ *is the image of* $\tilde{T}_{\mathscr{D}}$ *in* $\mathrm{End}_{o_{\mathscr{D}}} I_{(k,w),\mathscr{D},\chi}$*.*

(3) *The reduced Hecke algebra* $\tilde{T}^{\text{ß}}_{\mathscr{D}}$ *is the* $o_{\mathscr{D}}$*-subalgebra of* $\tilde{T}_{\mathscr{D}}$ *generated by* T_v *and* $T_{v,v}$ *for* $v \notin \Sigma_{\mathscr{D}}$*.* $\tilde{T}^{\text{ß}}_{\mathscr{D},\chi}$ *with a fixed central character* χ *is the image of* $\tilde{T}_{\mathscr{D}}$ *in* $\mathrm{End}_{o_{\mathscr{D}}} I_{(k,w),\mathscr{D},\chi}$

(4) *For any* $o_{\mathscr{D}}$*-algebra* R*,* $\tilde{T}_{\mathscr{D},R} = \tilde{T}_{\mathscr{D}} \otimes_{o_{\mathscr{D}}} R$*,* $\tilde{T}_{\mathscr{D},\chi,R} = \tilde{T}_{\mathscr{D},\chi} \otimes_{o_{\mathscr{D}}} R$*. We use the similar notation for the reduced Hecke algebras.*

By the definition, $\tilde{T}^{\text{ß}}_{\mathscr{D}}$ (resp. $\tilde{T}^{\text{ß}}_{\mathscr{D},\chi}$) is a subring of $\tilde{T}_{\mathscr{D}}$ (resp. $\tilde{T}_{\mathscr{D},\chi}$), and $\tilde{T}_{\mathscr{D},\chi}$ (resp. $\tilde{T}^{\text{ß}}_{\mathscr{D},\chi}$) is a quotient ring of $\tilde{T}_{\mathscr{D}}$ (resp. $\tilde{T}^{\text{ß}}_{\mathscr{D}}$).

We define a maximal ideal $m_{\mathscr{D}}$ of $\tilde{T}_{\mathscr{D}}$ corresponding to $\bar{\rho}$ as follows.

For a minimal lift $\pi_{\min}$, an o_{E_λ}-algebra homomorphism $f_{\pi_{\min}} : \tilde{T}^{\text{ß}}_{\mathscr{D}_{\min},\chi_{\bar{\rho}}} \to o_{E_\lambda}$ is defined. Here $\chi_{\bar{\rho}} = \chi^{\mathrm{Hecke}}_{\bar{\rho},w}$ is as in Definition 6.10. For $v \in P^{\mathbf{n.o.}}$, let $\mu_{\pi_{\min},v}$ be the nearly ordinary character of $\pi_{\min}$ at v (Definition 6.5), $\kappa_{\pi_{\min},v} = \mu_{\pi_{\min},v}|_{o^{\times}_{F_v}}$ the nearly ordinary type.

By $\tilde{m}_{\mathscr{D}}$ we denote the maximal ideal of $\tilde{T}_{\mathscr{D}}$ generated by $m_{o_{\mathscr{D}}}$ and the following operators:

- $T_v - f_{\pi_{\min}}(T_v),\ T_{v,v} - f_{\pi_{\min}}(T_{v,v})$ for $v \notin \Sigma_{\mathscr{D}}$.
- $U(p_v),\ U(p_v, p_v) - \chi_{\bar{\rho}}(p_v)$ for $v \in P^{\mathbf{u}}_{\mathscr{D}} \setminus P^{\mathrm{exc}}$.
- $\tilde{U}(p_v) - \alpha_{\pi_{\min},v},\ \tilde{U}(p_v, p_v) - \gamma_{\pi_{\min},v}$ for $v \in P^{\mathbf{n.o.}}$. $\alpha_{\pi_{\min},v} = (\prod_{\iota \in I_{F,v}} \iota(p_v)^{-k'_\iota}) \cdot \mu_{\pi_{\min},v}(p_v)$, $\gamma_{\pi_{\min},v} = q_v^{-w} \chi_{\bar{\rho}}(p_v)$.

Here, $\tilde{m}_{\mathscr{D},\chi}$ is the maximal ideal similarly defined for $\tilde{T}_{\mathscr{D},\chi}$.

Lemma 7.3. $\tilde{m}_{\mathscr{D}}$ *and* $\tilde{m}_{\mathscr{D},\chi_{\bar{\rho}}}$ *are proper ideals of* $\tilde{T}_{\mathscr{D}}$ *and* $\tilde{T}_{\mathscr{D},\chi_{\bar{\rho}}}$*, respectively.*

Proof of Lemma 7.3. It suffices to prove it for $\tilde{T}_{\mathscr{D},\chi_{\bar{\rho}}}$. For $\pi = \pi_{\min}$, we look at the intertwining space $I_{(k,w),\mathscr{D}}(\pi) = \otimes_{v \in |F|_f} I_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v)$, and calculate the action of generators of $\tilde{m}_{\mathscr{D},\chi_{\bar{\rho}}}$.

The central character of $\pi_{\min}$ is $\chi_{\bar{\rho}}$. $T_{v,v}$, $U(p_v, p_v)$, $\tilde{U}(p_v, p_v)$ are evidently compatible with this central character. So we only consider the actions of T_v, $U(p_v)$, and $\tilde{U}(p_v)$.

For $v \notin \Sigma_{\mathscr{D}}$, the action of T_v on $I(\bar{\rho}|_{F_v}, \pi_v)$ is $f_{\pi_{\min}}(T_v)$.

For $v \in P^{\mathbf{n.o.}}$, $\alpha_{\pi_{\min},v}$ is the $U(p_v)$-eigenvalue for a non-zero nearly ordinary vector, So the part in $I_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v)$ where $U(p_v)$ acts as $\alpha_{\pi_{\min},v}$ is non-zero.

For $v \in P^{\mathbf{u}}_{\mathscr{D}} \setminus P^{\mathrm{exc}}$, by Proposition 3.29 (2), $I_{\mathbf{u}}(\bar{\rho}|_{F_v}, \pi_v)$ has a non-zero subspace where $U(p_v)$ acts by zero map. So the claim follows. □

Definition 7.4. *Let* $\mathscr{D}$ *be a deformation type of* $\bar{\rho}$*.*

(1) *The* ℓ*-adic Hecke algebra* $T_{\mathscr{D}}$ *of* $\bar{\rho}$ *with deformation type* $\mathscr{D}$ *is*

$$T_{\mathscr{D}} = (\tilde{T}_{\mathscr{D}})_{\tilde{m}_{\mathscr{D}}},$$

and the ℓ*-adic Hecke algebra with a fixed central character* χ *is*

$$T_{\mathscr{D},\chi} = (\tilde{T}_{\mathscr{D},\chi})_{\tilde{m}_{\mathscr{D},\chi}}.$$

(2) *The* ℓ*-adic reduced Hecke algebra* $T^{\text{ß}}_{\mathscr{D}}$ *(resp.* $T^{\text{ß}}_{\mathscr{D},\chi}$*) is the image of* $\tilde{T}^{\text{ß}}_{\mathscr{D}}$ *(resp.* $\tilde{T}^{\text{ß}}_{\mathscr{D},\chi}$*) in* $T_{\mathscr{D}}$ *(resp.* $T_{\mathscr{D},\chi}$*).*

By the definition, $T^{\text{ß}}_{\mathscr{D}}$ is the $o_{\mathscr{D}}$-subalgebra of $T_{\mathscr{D}}$ generated by $T_v, T_{v,v}$ for $v \notin \Sigma_{\mathscr{D}}$, and the maximal ideal is generated by $m_{o_{\mathscr{D}}}$ and $T_v - f_{\pi_{\min}}(T_v),\ T_{v,v} - f_{\pi_{\min}}(T_{v,v})$ for $v \notin \Sigma_{\mathscr{D}}$.

Let $e_{\bar{\rho}}$ (resp. $e_{\bar{\rho},\chi}$) be the idempotent in $\tilde{T}_{\mathscr{D}}$ (resp. $\tilde{T}_{\mathscr{D},\chi}$) which defines the direct factor $T_{\mathscr{D}}$ (resp. $T_{\mathscr{D},\chi}$).

Definition 7.5. *The intertwining spaces which corresponds to $\bar{\rho}$ are defined by*

$$I_{(k,w),\mathscr{D}}(\bar{\rho}) = e_{\bar{\rho}} I_{(k,w),\mathscr{D}},$$

and

$$I_{(k,w),\mathscr{D},\chi}(\bar{\rho}) = e_{\bar{\rho}} I_{(k,w),\mathscr{D},\chi}.$$

Lemma 7.6. (1) *$I_{(k,w),\mathscr{D}}(\bar{\rho})$ is free of rank one as a $T_{\mathscr{D},\bar{E}_{\mathscr{D}}} = T_{\mathscr{D}} \otimes_{o_{\mathscr{D}}} \bar{E}_{\mathscr{D}}$-module. The same is true for $T_{\mathscr{D},\chi_{\bar{\rho}},\bar{E}_{\mathscr{D}}} = T_{\mathscr{D},\chi_{\bar{\rho}}} \otimes_{o_{\mathscr{D}}} \bar{E}_{\mathscr{D}}$.*
(2) *$T^{\text{ß}}_{\mathscr{D},E_{\mathscr{D}}} = T_{\mathscr{D},E_{\mathscr{D}}}$ and $T^{\text{ß}}_{\mathscr{D},\chi_{\bar{\rho}},E_{\mathscr{D}}} = T_{\mathscr{D},\chi_{\bar{\rho}},E_{\mathscr{D}}}$ hold.*

Proof of Lemma 7.6. We prove the lemma for $T_{\mathscr{D}}$. $T_{\mathscr{D},\chi_{\bar{\rho}}}$ is treated similarly.

First we prove (1). By Lemma 7.3, $I_{(k,w),\mathscr{D}}(\bar{\rho})$ is non-zero. For an element π of $\mathscr{A}_{(k,w)}(G_D)$ which appears in the decomposition of $I_{(k,w),\mathscr{D}}(\bar{\rho})$, we show that $e_{\bar{\rho}} I_{(k,w),\mathscr{D}}(\pi)$ is one dimensional over $\bar{E}_{\mathscr{D}}$.

$I_{(k,w),\mathscr{D}}(\pi) = \bigotimes_{v\in|F|_f} I_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v)$, where π_v the v-component of the restricted tensor product $\pi_f = \bigotimes_{v\in|F|_f} \pi_v$.

Take an ℓ-adic field $E'_{\lambda'}$ so that it contains $E_{\mathscr{D}}$ and π_f is defined over $E'_{\lambda'}$. The associated $E'_{\lambda'}$-representation $\rho' = \rho_{\pi,E'_{\lambda'}} : G_F \to \mathrm{GL}_2(o_{E'_{\lambda'}})$ is a deformation of $\bar{\rho}$: the mod λ'-reduction $\bar{\rho}' = \rho' \mod \lambda'$ has the same semi-simplification as $\bar{\rho}$ since the characteristic polynomial of Fr_v for $v \notin \Sigma_{\mathscr{D}}$ are the same, and hence isomorphic over $k_{\lambda'}$.

For $v \notin \Sigma_{\mathscr{D}}$, $I_{\mathbf{f}}(\bar{\rho}|_{F_v}, \pi_v)$ is one dimensional since π_v is spherical.

For $v \in P^{\mathbf{f}}_{\mathscr{D}}$, by Proposition 3.27, it is one dimensional, since π_v corresponds to $\rho'|_{F_v}$ which is a deformation of $\bar{\rho}$ by the compatibility of the local and the global Langlands correspondence.

For $v \in P^{\mathrm{exc}} \cap P^{\mathbf{u}}_{\mathscr{D}}$, $I_{\mathbf{u}}(\bar{\rho}|_{F_v}, \pi_v)$ is one dimensional by Proposition 3.29, (1).

For $v \in P^{\mathbf{u}}_{\mathscr{D}} \setminus P^{\mathrm{exc}}$, we need to consider $U(p_v)$-action on $I_{\mathbf{u}}(\bar{\rho}|_{F_v}, \pi_v)$. By Proposition 3.29, (2), it is isomorphic to $\bar{E}_{\lambda'}[U]/(U \cdot L(U,\ \pi_v \otimes \nu_v^{-1}))$, where U acts as $U(p_v)$, and ν_v is a character of $F_v^\times$ defined in Proposition 3.29. By our definition of $\tilde{m}_{\mathscr{D}}$, $U(p_v)$ acts as zero mod $\tilde{m}_{\mathscr{D}}$. Since the all roots of the local L-factor $L(U,\ \pi_v \otimes \nu_v^{-1})$ have ℓ-adic units, the localization at $(\lambda', U(p_v))$ of the $o_{E'_{\lambda'}}[U(p_v)]$-module $I_{\mathbf{u}}(\bar{\rho}|_{F_v}, \pi_v)$ is one dimensional, and this is the part which contributes to $e_{\bar{\rho}} I_{(k,w),\mathscr{D}}(\pi)$.

For $v \in P^{\mathbf{fl}}$, $I_{\mathbf{fl}}(\bar{\rho}|_{F_v}, \pi_v)$ is one dimensional since π_v is spherical, and $K(\bar{\rho}|_{F_v}) = \mathrm{GL}_2(o_{F_v})$.

For $v \in P^{\mathbf{n.o.}}$, the argument is similar to the case of $P^{\mathbf{u}}_{\mathscr{D}}$, using Proposition 6.15 instead, by specifying the one dimensional subspace where $\tilde{U}(p_v)$-operator acts by a scalar multiplication by an ℓ-adic unit.

Since the action of $T_{\mathscr{D},\bar{E}_{\mathscr{D}}}$ on $I_{(k,w),\mathscr{D}}(\bar{\rho})$ is faithful, the one-dimensionality of $e_{\bar{\rho}} I_{(k,w),\mathscr{D}}(\pi)$ shows that $T_{\mathscr{D}}$ is reduced. It is easy to see that any $\bar{E}_{\mathscr{D}}$-homomorphism $f : T_{\mathscr{D},\bar{E}_{\mathscr{D}}} \to \bar{E}_{\mathscr{D}}$ is obtained in this way, and (1) is shown. (2) follows from (1) since $T^{\text{ß}}_{\mathscr{D}}$-action on each non-zero component $e_{\bar{\rho}} I_{(k,w),\mathscr{D}}(\pi)$ is non-trivial. $\square$

Corollary 7.7. *For any element $\pi \in \mathscr{A}_{(k,w)}(G_{\bar{\rho}})$, $e_{\bar{\rho}} I_{(k,w),\mathscr{D}}(\pi)$ is isomorphic to $\bigotimes_{v\in|F|_f} I'_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v)$. Here, $I'_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v)$ is the subspace of $I_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v)$ defined as follows.*

- *For $v \notin P^{\mathbf{n.o.}} \cup P^{\mathbf{u}}_{\mathscr{D}}$, $I'_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v) = I_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v)$.*
- *For $v \in P^{\mathbf{u}}_{\mathscr{D}}$, $I'_{\mathbf{u}}(\bar{\rho}|_{F_v}, \pi_v)$ is the subspace of $I_{\mathbf{u}}(\bar{\rho}|_{F_v}, \pi_v)$ where $U(p_v)$ acts by zero map.*
- *For $v \in P^{\mathbf{n.o.}}$, $I'_{\mathrm{def}_{\mathscr{D}}(v)}(\bar{\rho}|_{F_v}, \pi_v)$ is the $U(p_v)$-eigensubspace of $I_{\mathbf{u}}(\bar{\rho}|_{F_v}, \pi_v)$ of the unique ℓ-adic unit eigenvalue.*

This immediately follows from the proof of Lemma 7.6.

7.3. **Galois deformations and ℓ-adic Hecke algebras.** We have defined ℓ-adic Hecke algebras $T_{\mathscr{D}}^{ß} \subset T_{\mathscr{D}}$. By the existence of the ℓ-adic representation attached to the elements of $\mathscr{A}_{(k,w)}(G_{\bar{\rho}})$ (cf. §4.4), there exists a continuous representation

$$\rho_{\mathscr{D},E_\lambda}^{\mathrm{mod}} : G_{\Sigma_{\mathscr{D}}} \longrightarrow \mathrm{GL}_2(T_{\mathscr{D},E_\lambda}^{ß})$$

such that

$$\begin{cases} \mathrm{trace}\rho_{\mathscr{D},E_\lambda}^{\mathrm{mod}}(\mathrm{Fr}_v) &= T_v \\ \det\rho_{\mathscr{D},E_\lambda}^{\mathrm{mod}}(\mathrm{Fr}_v) &= q_v \cdot T_{v,v} \quad \text{for each } v \notin \Sigma_{\mathscr{D}}. \end{cases}$$

Lemma 7.8. *For $T_{\mathscr{D}}^{ß}$ attached to a deformation type $\mathscr{D}$, the following holds:*

(1) $T_{\mathscr{D}}^{ß}$ *contains* $\mathrm{trace}_{T_{\mathscr{D},E_{\mathscr{D}}}^{ß}} \rho_{\mathscr{D},E_\lambda}^{\mathrm{mod}}(\sigma)$ *for* $\sigma \in G_{\Sigma_{\mathscr{D}}}$.

(2) *For a finite set of finite places Σ which contains $\Sigma_{\mathscr{D}}$, $T_{\mathscr{D}}^{ß}$ is generated by T_v, $v \notin \Sigma$ as an $o_{\mathscr{D}}$-algebra.*

Proof of Lemma 7.8. (Cf. [51].) Let $\tilde{T}$ be the normalization of $T_{\mathscr{D}}^{ß}$, T_Σ the $o_{\mathscr{D}}$-subalgebra of $T_{\mathscr{D}}^{ß}$ generated by T_v, $v \notin \Sigma$, T' the $o_{\mathscr{D}}$-subalgebra of $\tilde{T}$ generated by $\mathrm{trace}\rho_{\mathscr{D},E_\lambda}^{\mathrm{mod}}(\sigma), \sigma \in G_{\Sigma_{\mathscr{D}}}$. It suffices to see the images of T_Σ and T' in $\tilde{T}/\lambda^n\tilde{T}$ is the same for any $n \geq 1$. $\rho_{\mathscr{D},E_\lambda}^{\mathrm{mod}}$ is defined over $\tilde{T}$, which we denote by $\tilde{\rho}$. By the Chebotarev density theorem, for any $\sigma \in G_{\Sigma_{\mathscr{D}}}$, there is a finite place $v \notin \Sigma$ such that $\tilde{\rho}(\mathrm{Fr}_v)$ and $\tilde{\rho}(\sigma)$ is conjugate in $\tilde{T}/\lambda^n\tilde{T}$. This means that $\mathrm{trace}_{\tilde{T}}\tilde{\rho}(\sigma) \equiv \mathrm{trace}_{\tilde{T}}\tilde{\rho}(\mathrm{Fr}_v) \mod \lambda^n$, and $T_\Sigma = T'$. The equality $2q_vT_{v,v} = (\mathrm{trace}\rho(\mathrm{Fr}_v))^2 - \mathrm{trace}\rho(\mathrm{Fr}_v^2)$ for $v \notin \Sigma_{\mathscr{D}}$ implies that $T' = T_{\mathscr{D}}^{ß}$. □

Proposition 7.9. *(Functoriality) For two deformation types $\mathscr{D}$ and $\mathscr{D}'$ of $\bar{\rho}$, assume that there is a morphism $\mathscr{D} \to \mathscr{D}'$ in* $\mathrm{Type}(\bar{\rho})$ *(Definition 3.32). Then there is a canonical surjective homomorphism $T_{\mathscr{D}'} \to T_{\mathscr{D}} \otimes_{o_{\mathscr{D}}} o_{\mathscr{D}'}$.*

This statement is non-trivial: T_v and $T_{v,v}$-operators are missing in $T_{\mathscr{D}'}$ when $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{f}$, and $\mathrm{def}_{\mathscr{D}'}(v) = \mathbf{u}$. By Lemma 7.8, these missing operators are recovered from $\rho_{\mathscr{D},E_\lambda}^{\mathrm{mod}}$.

Proposition 7.10. *Assume that the residual representation $\bar{\rho}$ of $\rho_{\mathscr{D},E_{\mathscr{D}}}^{\mathrm{mod}} : G_{\Sigma_{\mathscr{D}}} \to \mathrm{GL}_2(T_{\mathscr{D},E_{\mathscr{D}}}^{ß})$ is irreducible. Then there exists a representation*

$$\rho_{\mathscr{D}}^{\mathrm{mod}} : G_{\Sigma_{\mathscr{D}}} \to \mathrm{GL}_2(T_{\mathscr{D}}^{ß})$$

having the same trace and determinant as $\rho_{\mathscr{D},E_\lambda}^{\mathrm{mod}}$.

Proof of Proposition 7.10. Take a basis of $(T_{\mathscr{D},E_{\mathscr{D}}}^{ß})^{\oplus 2}$ such that

$$\rho_{\mathscr{D},E_{\mathscr{D}}}^{\mathrm{mod}}(c) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here, c is a complex conjugation. Then the entries of

$$\rho_{\mathscr{D},E_{\mathscr{D}}}^{\mathrm{mod}}(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$$

have the property that $b(\sigma) \cdot c(\sigma)$ is contained in $T_{\mathscr{D}}^{ß}$ and independent of a choice of basis. Using the irreducibility of $\bar{\rho}$, one shows the existence of some τ such that $b(\tau)c(\tau)$ is a unit. Change the basis again so that $b(\tau) = 1$. Then $\rho_{\mathscr{D},E_{\mathscr{D}}}^{\mathrm{mod}}$ is defined over $T_{\mathscr{D}}^{ß}$ using this basis, since

$$\begin{cases} a(\sigma) = 1/2(\mathrm{trace}\rho_{\mathscr{D},E_{\mathscr{D}}}^{\mathrm{mod}}(\sigma) + \mathrm{trace}\rho_{\mathscr{D},E_{\mathscr{D}}}^{\mathrm{mod}}(\sigma \cdot c)) \\ d(\sigma) = 1/2(\mathrm{trace}\rho_{\mathscr{D},E_{\mathscr{D}}}^{\mathrm{mod}}(\sigma) - \mathrm{trace}\rho_{\mathscr{D},E_{\mathscr{D}}}^{\mathrm{mod}}(\sigma \cdot c)) \\ b(\sigma) = x(\sigma,\tau)/x(\tau,\tau) \\ c(\sigma) = x(\tau,\sigma). \end{cases}$$

Here $x(\sigma,\tau) = a(\sigma\tau) - a(\sigma)a(\tau) = b(\sigma)c(\tau)$. So the claim follows. $\square$

The following lemma is proved in a similar way, so the proof is omitted.

Lemma 7.11. *Let G be a group with $c \in G$, $c^2 = 1$, R a local ring such that 2 is a non-zero divisor, $\rho, \rho' : G \to \mathrm{GL}_2(R)$ two representations with the irreducible residual representations such that $\det\rho(c) = \det\rho'(c) = -1$. $\mathrm{trace}\rho(g) = \mathrm{trace}\rho'(g)$ for any $g \in G$, then ρ and ρ' are isomorphic.*

By Lemma 7.8, $\rho^{\mathrm{mod}}_{\mathscr{D}}$ is unique up to isomorphisms.

7.4. **Cohomology groups of modular varieties as Hecke modules.** We construct $T_{\mathscr{D}}$-module $M_{\mathscr{D}}$ from a cohomology group of the modular variety associated to the deformation type $\mathscr{D}$. First we show how to go around the technical difficulty we encounter when a compact open subgroup is not small by the method of auxiliary places.

For a given absolutely irreducible $\bar\rho$, a deformation type $\mathscr{D}$, there are infinitely many finite places y which satisfies the following properties:

- $y \notin \Sigma_{\mathscr{D}}$, and $q_y \geq a_F$, where a_F is the integer given in Lemma 4.11.
- $q_y \not\equiv 1 \mod \ell$, and the eigenvalues $\bar\alpha_y$, $\bar\beta_y$ of $\bar\rho(\mathrm{Fr}_y)$ counted with the multiplicities satisfy $\bar\alpha_y \neq q_y^{\pm 1}\bar\beta_y$.

Note that we do not exclude the case when $\bar\alpha_y$ equals $\bar\beta_y$.

The existence of finite places with these two properties follows from [13], lemma 11 (in the reference, the linear independence of F and $\mathbb{Q}(\zeta_\ell)$ is assumed, which can be removed by a slight modification. See also [15] for an argument which is independent of the classification of subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$).

For an auxiliary place y that satisfies the conditions as above, define the deformation function $\mathrm{def}_{\mathscr{D}_y}$ by

$$\mathrm{def}_{\mathscr{D}_y}(v) = \mathrm{def}_{\mathscr{D}}(v) \text{ if } v \neq y,\ \mathrm{def}_{\mathscr{D}_y}(y) = \mathbf{u},$$

and let $\mathscr{D}_y$ be the deformation type which has the defomation function $\mathrm{def}_{\mathscr{D}_y}$ as above, and the same data as $\mathscr{D}$ except for the deformation function.

There is a natural surjective homomorphism $T_{\mathscr{D}_y} \twoheadrightarrow T_{\mathscr{D}}$ by Proposition 7.9. By the conditions imposed on y, this is in fact an isomorphism:

Proposition 7.12. *There is a natural $o_{\mathscr{D}}$-algebra isomorphism*

$$T_{\mathscr{D}_y} \simeq T_{\mathscr{D}}.$$

The same is true for $T_{\mathscr{D}_y,\chi_{\bar\rho}}$ and $T_{\mathscr{D},\chi_{\bar\rho}}$.

Proof of Proposition 7.12. We prove the claim for $T_{\mathscr{D}_y}$. The case of $T_{\mathscr{D}_y,\chi_{\bar\rho}}$ follows from it. Take an element π of $\mathscr{A}_{(k,w)}(G_D)$ which contributes to $I_{(k,w),\mathscr{D}_y}(\bar\rho)$ non-trivially. We show that π appears in $I_{(k,w),\mathscr{D}}(\bar\rho)$. By enlarging $E_{\mathscr{D}} = E_{\mathscr{D}_y}$ if necessary, we may assume that π_f is defined over $E_{\mathscr{D}}$.

The Galois representation $\rho = \rho_{\pi,E_{\mathscr{D}}} : G_F \to \mathrm{GL}_2(o_{E_{\mathscr{D}}})$ attached to π is a deformation of $\bar\rho$, and the local representation $\rho|_{F_y}$ corresponds to the y-component π_y of π by the compatibility of the local and the global Langlands correspondences.

If π_y is supercuspidal, $\rho_y = \rho|_{F_y}$ must be induced from a tame quadratic extension $\tilde F_y$ of F_y: $\rho_y|_{G_{\tilde F_y}} \simeq \chi \oplus \chi'$, χ' is the twist of χ by the non-trivial element in $\mathrm{Gal}(\tilde F_y/F_y)$.

χ/χ' must have an ℓ-power order, otherwise any mod $m_{o_{\mathscr{D}}}$ reduction of ρ_y with respect to an $o_{\mathscr{D}}$-lattice is absolutely irreducible. Thus we have

$$(*) \qquad (\bar\rho_y)^{\mathrm{ß}} = \bar\chi \otimes \mathrm{Ind}_{G_{\tilde F_y}}^{G_{F_y}} 1 \simeq \bar\chi \oplus \bar\chi \cdot \chi_{\tilde F_y/F_y}.$$

Here $\bar{\chi}$ is the mod $m_{o_{\mathscr{D}}}$ reduction of χ, and $\chi_{\tilde{F}_y/F_y}$ is the quadratic character of G_{F_y} which corresponds to $\tilde{F}_y/F_y$. Since $\bar{\rho}_y$ is unramified, $\tilde{F}_y$ is the unramified quadratic extension of F_y, which implies that $q_y \equiv -1 \mod \ell$. $(*)$ implies that the two eigenvalues of $\bar{\rho}_y$ are of the form $\bar{\alpha}$, $-\bar{\alpha}$, and this contradicts to our choice of y since $q_y \equiv -1 \mod \ell$.

Since the Swan conductor remains the same under a mod $m_{o_{\mathscr{D}}}$-reduction, ρ_y is a tame representation, and hence π_y is a special representation twisted by a tame character, or a tame principal series representation. In the former case, the two eigenvalues of any frobenius lift is of the form α, $q_y\alpha$, which is a contradiction. Since $q_y \not\equiv 1 \mod \ell$, and $\bar{\rho}_y$ is unramified, π_y is spherical. The eigenvalues of T_y and $T_{y,y}$-operators on $\pi_y^{K_{\mathscr{D},y}}$ are congruent to trace$\bar{\rho}(\mathrm{Fr}_y)$ and $q_y^{-1}\det\bar{\rho}(\mathrm{Fr}_y)$ modulo $m_{o_{\mathscr{D}}}$, thus π must contribute to $I_{(k,w),\mathscr{D}}(\bar{\rho})$. □

For an element $\pi \in \mathscr{A}_{(k,w)}(G_{\bar{\rho}})$, we modify the intertwining space $I_{(k,w),\mathscr{D}_y}$ and $I_{(k,w),\mathscr{D}_y}(\pi)$ to

$$I^y_{(k,w),\mathscr{D}_y} = \mathrm{Hom}_{K_{\mathscr{D}}(y)}(\nu_{\mathscr{D}_y}|_{K_{\mathscr{D}}(y)}, \mathscr{S}_{(k,w)}(G_{\bar{\rho}})),$$

and

$$I^y_{(k,w),\mathscr{D}_y}(\pi) = \mathrm{Hom}_{K_{\mathscr{D}}(y)}(\nu_{\mathscr{D}_y}|_{K_{\mathscr{D}}(y)}, \pi_f).$$

$I^y_{(k,w),\mathscr{D}_y}$ (resp. $I^y_{(k,w),\mathscr{D}_y}(\pi)$) contains $I_{(k,w),\mathscr{D}_y}$ (resp. $I_{(k,w),\mathscr{D}_y}(\pi)$) as a subspace. The following lemma shows that the part corresponding to $\bar{\rho}$ is the same for $I^y_{(k,w),\mathscr{D}_y}$ and $I_{(k,w),\mathscr{D}_y}$.

Lemma 7.13. *For an element $\pi \in \mathscr{A}_{(k,w)}(G_{\bar{\rho}})$ which is defined over E_λ, if $I^y_{(k,w),\mathscr{D}_y}(\pi) \neq \{0\}$ and ρ_{π,E_λ} is a deformation of $\bar{\rho}$, then $I^y_{(k,w),\mathscr{D}_y}(\pi) = I_{(k,w),\mathscr{D}_y}(\pi)$.*

Proof of Lemma 7.13. As in the argument of the proof of Proposition 7.12, the y-component π_y of π is spherical, so it suffices to see $\pi_y^{K_{11}(m_y^2)} = \pi_y^{K_1(m_y^2)}$. Since the central character of π_y is unramified, the action of central elements $o_{F_y}^\times \cdot 1_{D(\bar{\rho})_{F_y}}$ is trivial. The claim follows. □

By Lemma 7.13, we regard $T_{\mathscr{D}_y}$ as the Hecke algebra acting on $I^y_{(k,w),\mathscr{D}_y}$.

We define a $T_{\mathscr{D}}$-module $M_{\mathscr{D}}$ for deformation type $\mathscr{D}$.

We modify the y-component of $K_{\mathscr{D}_y}$, and use $\tilde{K}_{\mathscr{D}_y} = K_{11}(m_y^2) \cdot K^y_{\mathscr{D}}$. Fix a finite set of finite places P_0. We choose a finite set S which satisfies the condition of Lemma 4.11 for $K = \tilde{K}_{\mathscr{D}_y}$ and P_0. By Proposition 4.9, $\tilde{K}_{\mathscr{D}_y}(y,S) = K_{11}(m_y^2) \cdot \prod_{u\in S} U_u \cdot K^{\{y\}\cup S}_{\mathscr{D}_y}$ is small, and $H^q_{\mathrm{stack}}(S_K\ \bar{\mathscr{F}}^K_{(k,w),R})$ is defined for any compact open subgroup K of $\tilde{K}_{\mathscr{D}_y}$ and a finite $o_{\mathscr{D}}$-algebra R (cf. §4.5).

For $\Sigma = \Sigma_{\mathscr{D}_y} \cup S$ and $K = \ker \nu_{\mathscr{D}_y}|_{\tilde{K}_{\mathscr{D}_y}}$, the convolution algebra $H_{K^\Sigma} = H(G_D(\mathbb{A}_{\mathbb{Q},f}), K^\Sigma)_{o_{\mathscr{D}}}$ acts on $H^q_{\mathrm{stack}}(S_K\ \bar{\mathscr{F}}^K_{(k,w),R})$. Let $\tilde{m}_{\Sigma,\bar{\rho}}$ be the maximal ideal of H_{K^Σ} which corresponds to $\bar{\rho}$ (cf. §5.1). By Proposition 5.5, the localization $H^q_{\mathrm{stack}}(S_K\ \bar{\mathscr{F}}^K_{(k,w)})_{\tilde{m}_{\Sigma,\bar{\rho}}}$ at $\tilde{m}_{\Sigma,\bar{\rho}}$ vanishes unless $q = q_{\bar{\rho}}$, where $q_{\bar{\rho}} = q_{D(\bar{\rho})}$. $H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_K\ \bar{\mathscr{F}}^K_{(k,w)})_{\tilde{m}_{\Sigma,\bar{\rho}}}$ is $o_{\mathscr{D}_y}$-free, and the formation commutes with scalar extensions.

Definition 7.14. *We define $\tilde{M}^y_{\mathscr{D}_y}$ by*

$$\tilde{M}^y_{\mathscr{D}_y} = \mathrm{Hom}_{\tilde{K}_{\mathscr{D}_y}}(\nu_{\mathscr{D}_y}|_{\tilde{K}_{\mathscr{D}_y}}, H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_K, \bar{\mathscr{F}}^K_{(k,w)})_{\tilde{m}_{\Sigma,\bar{\rho}}}).$$

Since the order of $\nu_{\mathscr{D}_y}$ is prime to ℓ, $\tilde{M}^y_{\mathscr{D}_y}$ is regarded as an $o_{\mathscr{D}_y}$-direct summand of $H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_K, \bar{\mathscr{F}}^K_{(k,w)})$ where $\tilde{K}_{\mathscr{D}_y}$ acts by $\nu_{\mathscr{D}_y}$.

For $\tilde{M}^y_{\mathscr{D}_y}$ defined as above, the following proposition holds by the decomposition in Proposition 4.4.

Proposition 7.15. *We have*

$$\tilde{M}^y_{\mathscr{D}_y} \otimes_{o_{\mathscr{D}_y}} \bar{E}_{\mathscr{D}_y} = \bigoplus_{\pi \in \mathscr{A}_{\bar{\rho}}} I^y_{(k,w),\mathscr{D}_y}(\pi)^{\oplus 2^{q_{\bar{\rho}}}},$$

where $\mathscr{A}_{\bar{\rho}}$ is the subset of $\mathscr{A}_{(k,w)}(G_D)$ consisting of the representations π whose associated Galois representation $\rho_{\pi,\bar{E}_{\mathscr{D}_y}}$ is a deformation of $\bar{\rho}$.

As $\bar{\rho}$ is not of residual type, the contribution from $\mathscr{A}^c_{(k,w)}(G_D)$ disappears. The standard Hecke operators $T_v, T_{v,v}$ for $v \notin \Sigma$ and $\tilde{U}(p_v), \tilde{U}(p_v, p_v)$-operators for $v \in \Sigma_{\mathscr{D}_y} \setminus P^{\mathrm{exc}}$ acts on $\tilde{M}^y_{\mathscr{D}_y}$. By Lemma 7.8 (2), the image of H_{K^Σ} in $\mathrm{End}_{o_{\mathscr{D}_y}} \tilde{M}^y_{\mathscr{D}_y}$ is $T^{ß}_{\mathscr{D}_y}$ (the missing T_v and $T_{v,v}$-operators for $v \in S$ are recovered from $\rho^{\mathrm{mod}}_{\mathscr{D}_y}$). Let $\tilde{T}^y_{\mathscr{D}_y}$ be the $T^{ß}_{\mathscr{D}_y}$-algebra generated by $U(p_v), U(p_v, p_v)$ for $v \in P^{\mathbf{u}}_{\mathscr{D}_y} \setminus P^{\mathrm{exc}}$ and $\tilde{U}(p_v), \tilde{U}(p_v, p_v)$ for $v \in P^{\mathbf{n.o.}}$ in $\mathrm{End}_{o_{\mathscr{D}_y}} \tilde{M}^y_{\mathscr{D}_y}$. If we define the maximal ideal $\tilde{m}^y_{\mathscr{D}_y}$ of $\tilde{T}_{\mathscr{D}_y}(y)$ similarly as in §7.2, by Lemma 7.13, the localization of $\tilde{T}^y_{\mathscr{D}_y}$ at $\tilde{m}^y_{\mathscr{D}_y}$ is identified with $\tilde{T}_{\mathscr{D}_y}$.

Definition 7.16. *$M_{\mathscr{D}}$ is a $\tilde{T}_{\mathscr{D}_y}$-module given by*

$$M_{\mathscr{D}} = (\tilde{M}^y_{\mathscr{D}_y})_{\tilde{m}^y_{\mathscr{D}_y}}.$$

Note that $M_{\mathscr{D}}$ is a $T_{\mathscr{D}}$-module by the identification $T_{\mathscr{D}} \simeq T_{\mathscr{D}_y}$ given in Proposition 7.12, and $M_{\mathscr{D},E_{\mathscr{D}}} = M_{\mathscr{D}} \otimes_{o_{\mathscr{D}}} E_{\mathscr{D}}$ is free of rank $2^{q_{\bar{\rho}}}$ over $T_{\mathscr{D},E_{\mathscr{D}}}$ by the decomposition in Proposition 7.15, Lemma 7.6, and Lemma 7.13.

Remark 7.17. *It is possible to show that $M_{\mathscr{D}}$ is independent of a choice of auxiliary places y and S up to isomorphisms.*

The following theorem, which is proved in the next subsections, is the main result in this section.

Theorem 7.18. *Assume that $\bar{\rho}$ is an irreducible modular representation of type $\mathscr{D}$, and the validity of Hypothesis 6.7 for the representations π of $G_D(\mathbb{A}_{\mathbb{Q},f})$ which contributes to $T^{ß}_{\mathscr{D},\bar{E}_{\mathscr{D}}}$. Then we have the following:*

(1) *The representation $\rho^{\mathrm{mod}}_{\mathscr{D}}$ is a deformation of $\bar{\rho}$ of type $\mathscr{D}$.*
(2) *$T^{ß}_{\mathscr{D}} = T_{\mathscr{D}}$.*

Assume the validity of this theorem for the moment. Let $R_{\mathscr{D}}$ be the universal deformation ring of $\bar{\rho}$ of deformation type $\mathscr{D}$. By the universality of $R_{\mathscr{D}}$, we have the $o_{\mathscr{D}}$-algebra homomorphism

$$\pi_{\mathscr{D}} : R_{\mathscr{D}} \longrightarrow T_{\mathscr{D}}$$

which corresponds to $\rho^{\mathrm{mod}}_{\mathscr{D}}$ by Theorem 7.18, (1). $\pi_{\mathscr{D}}$ is surjective by Theorem 7.18, (2). So $M_{\mathscr{D}}$ is regarded as an $R_{\mathscr{D}}$-module, and we will construct a Taylor-Wiles system for the pair $(R_{\mathscr{D}}, M_{\mathscr{D}})$ in the next section.

7.5. **Local conditions outside ℓ.** In this subsection, we prove Theorem 7.18 at a finite place $v \nmid \ell$. If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{u}$, (1) is clear. For (2), we show that $U(p_v)$ and $U(p_v, p_v)$ belong to $T^{ß}_{\mathscr{D}}$. The $U(p_v)$-operator is zero. $U(p_v, p_v) = q_v^{-1} \det \rho^{\mathrm{mod}}_{\mathscr{D}}(p_v)$ by the identification $G^{\mathrm{ab}}_{F_v} \simeq \widehat{F_v^\times}$ by the local class field theory.

When $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{f}$, (2) is clear, so we show that $\rho^{\mathrm{mod}}_{\mathscr{D}}$ is a finite deformation of $\bar{\rho}$ at v.

Take an element π of $\mathscr{A}_{(k,w)}(G_D)$ which appears in the component of $T^{ß}_{\mathscr{D},\bar{E}_{\mathscr{D}}}$. By extending $E_{\mathscr{D}}$ if necessary, we may assume that π_f is defined over $E_{\mathscr{D}}$, and the associated Galois representation $\rho_{\pi,E_{\mathscr{D}}} : G_F \to \mathrm{GL}_2(o_{\mathscr{D}})$ is a deformation of $\bar{\rho}$.

By Corollary 7.7, $e_{\bar\rho}I_{(k,w),\mathscr{D}}(\pi)$ is isomorphic to $\otimes_{u\in|F|_f}I'_{\mathrm{def}_{\mathscr{D}}(u)}(\bar\rho|_{F_u},\pi_u)$, and $I'_{\mathrm{def}_{\mathscr{D}}}(\bar\rho|_{F_u},\pi_u) = I_{\mathrm{def}_{\mathscr{D}}(u)}(\bar\rho|_{F_u},\pi_u)$ if $\mathrm{def}_{\mathscr{D}}(u)=\mathbf{f}$. By the compatibility of the local and the global Langlands correspondences, π_v corresponds to $\rho_{\pi,v}=\rho_{\pi,E_{\mathscr{D}}}|_{F_v}$, which is a deformation of $\bar\rho|_{F_v}$. Since we assume that $e_{\bar\rho}I_{(k,w),\mathscr{D}}(\pi)$ is non-zero, $I_{\mathbf{f}}(\bar\rho|_{F_v},\pi_v)$ is non-zero. By Proposition 3.27, $\rho_{\pi,v}$ is a finite deformation of $\bar\rho|_{F_v}$.

If $\bar\rho|_{F_v}$ is of type 0_{NE}, we need to show that $\det\rho^{\mathrm{mod}}_{\mathscr{D}}$ is unramified at v (see Definition 3.1). It suffices to check this over $E_{\mathscr{D}}$, and for any element π of $\mathscr{A}_{(k,w)}(G_D)$ that contributes to $e_{\bar\rho}I_{(k,w),\mathscr{D}}$, $\det\rho_{\pi,v}$ is unramified since it is a finite deformation of $\bar\rho|_{F_v}$ (Definition 3.1).

If $\bar\rho|_{F_v}$ is of type 0_E, it is of the form $\mathrm{Ind}_{G_{\tilde F_v}}^{G_{F_v}}\bar\psi$ for the unramified quadratic extension $\tilde F_v$ of F_v. We must show that $\rho^{\mathrm{mod}}_{\mathscr{D}}|_{I_{\tilde F_v}}$ is isomorphic to the sum of $\psi|_{I_{\tilde F_v}}$ and the Frobenius twist (cf. Definition 3.1). Here ψ is the Teichmüller lift of $\bar\psi$. Again it suffices to see this over $E_{\mathscr{D}}$, and the claim follows since $\rho_{\pi,v}$ is a finite deformation of $\bar\rho|_{F_v}$ for any element π which appears in the component of $T^{\text{ß}}_{\mathscr{D},\bar E_{\mathscr{D}}}$.

If $\bar\rho|_{F_v}$ is absolutely reducible with the twist type $\bar\kappa_v$, we need to show that the I_{F_v}-fixed part $\mathscr{L}_v$ of $\rho^{\mathrm{mod}}_{\mathscr{D}}|_{I_{F_v}}\otimes\kappa_v^{-1}$ is $T^{\text{ß}}_{\mathscr{D}}$-free, is a $T^{\text{ß}}_{\mathscr{D}}$-direct summand, and the $T^{\text{ß}}_{\mathscr{D}}$-rank of $\mathscr{L}_v$ is equal to $d_v=\dim_k(\bar\rho|_{I_{F_v}}\otimes\bar\kappa^{-1}_{\mathscr{D},v})^{I_{F_v}}$. By the same argument as the previous cases, $\mathscr{L}_v\otimes_{o_{\mathscr{D}}}E_{\mathscr{D}}$ is $T^{\text{ß}}_{\mathscr{D},E_{\mathscr{D}}}$-free of rank d_v. If $d_v=2$, that is, in the case of 2_{PR}, it follows that $\mathscr{L}_v=(T^{\text{ß}}_{\mathscr{D}})^{\oplus 2}$, and the claim is shown.

So the case of $d_v=1$ is left. It is clear that $\mathscr{L}_v$ is an $o_{\mathscr{D}}$-direct summand of $(T^{\text{ß}}_{\mathscr{D}})^{\oplus 2}$, and G_{F_v}-stable as a subrepresentation of $\rho^{\mathrm{mod}}_{\mathscr{D}}|_{F_v}$. Thus we have an exact sequence of $G_{F_v}\times T^{\text{ß}}_{\mathscr{D}}$-modules

$$0\longrightarrow\mathscr{L}_v\longrightarrow\rho^{\mathrm{mod}}_{\mathscr{D}}|_{F_v}\longrightarrow\mathscr{L}'_v\longrightarrow 0,$$

where $\mathscr{L}'_v$ is $o_{\mathscr{D}}$-free, and has the $T^{\text{ß}}_{\mathscr{D}}$-rank one. The I_{F_v}-action on $\mathscr{L}'_v$ is $\det\rho^{\mathrm{mod}}_{\mathscr{D}}|_{I_{F_v}}\otimes\kappa^{-1}_{\mathscr{D},v}$ by calculating it over $E_{\mathscr{D}}$.

$\mathscr{L}'_v/m_{\mathscr{D}}\mathscr{L}'_v$ is a quotient of $\bar\rho$, and the G_{F_v}-action on $\mathscr{L}'_v/m_{\mathscr{D}}\mathscr{L}'_v$ is $\det\bar\rho|_{F_v}\otimes\bar\kappa_v^{-1}$. Since $\bar\rho|_{F_v}$ has a unique one dimensional quotient where G_{F_v} acts via $\det\bar\rho|_{F_v}\otimes\bar\kappa_v^{-1}$, $\dim_k\mathscr{L}'_v/m_{\mathscr{D}}\mathscr{L}'_v=1$. By Nakayama's lemma, there is a surjection $T^{\text{ß}}_{\mathscr{D}}\twoheadrightarrow\mathscr{L}'_v$. Since the rank of $\mathscr{L}'_v$ over $E_{\mathscr{D}}$ is one, $\mathscr{L}'_v$ is $T^{\text{ß}}_{\mathscr{D}}$-free of rank one, and hence $\mathscr{L}_v$ is also $T^{\text{ß}}_{\mathscr{D}}$-free of rank one. This shows that $\rho^{\mathrm{mod}}_{\mathscr{D}}$ is a finite deformation at v.

7.6. **Local conditions at ℓ.** We prove Theorem 7.18 at $v|\ell$ when $q_{\bar\rho}$ is one, under the Hypothesis 6.7 if $\mathrm{def}_{\mathscr{D}}(v)=\mathbf{n.o.}$.

We denote the residual characteristic at v by p. To emphasize the difference with the previous cases, we mainly use p instead of ℓ.

Assume that $\mathrm{def}_{\mathscr{D}}(v)=\mathbf{n.o.}$. Let $(k_v,w)=((k_\iota)_{\iota\in I_{F,v}},w)$ be the v-type, and $k'_v=(k'_\iota)_{\iota\in I_{F,v}}$ is as in Definition 6.1.

By Hypothesis 6.7, for any representation $\pi\in\mathscr{A}_{(k,w)}(G_{\bar\rho})$ which appear as a component of $T^{\text{ß}}_{\mathscr{D},E_{\mathscr{D}}}$, $\rho_{\pi,E_{\mathscr{D}}}|_{F_v}$, contains a subrepresentation $L_{k'_v}(\chi_v)$ with multiplicity one. Here χ_v is the nearly ordinary character of π at v, and $\chi_v|_{o^\times_{F_v}}$ is the automorphic nearly ordinary type of $\bar\rho$ at v by Definition 6.14. By choosing this subrepresentation for each π, we may assume that $\rho^{\mathrm{mod}}_{\mathscr{D},E_{\mathscr{D}}}|_{F_v}$ has the form

$$0\longrightarrow\mathscr{L}_{E_\lambda}\longrightarrow\rho^{\mathrm{mod}}_{\mathscr{D},E_{\mathscr{D}}}|_{F_v}\longrightarrow\mathscr{L}'_{E_\lambda}\longrightarrow 0,$$

where $\mathscr{L}_{E_\lambda}$, $\mathscr{L}'_{E_\lambda}$ are both free of rank one over $T_{\mathscr{D},E_{\mathscr{D}}}$. By the same argument as in the case of $d_v=1$ of §7.5 using that $\bar\rho|_{F_v}$ is G_{F_v}-distinguished, $\mathscr{L}=(T^{\text{ß}}_{\mathscr{D}})^{\oplus 2}\cap\mathscr{L}_{E_{\mathscr{D}}}$ and $\mathscr{L}'=(T^{\text{ß}}_{\mathscr{D}})^{\oplus 2}/\mathscr{L}$ are free of rank one over $T^{\text{ß}}_{\mathscr{D}}$. This shows that $\rho^{\mathrm{mod}}_{\mathscr{D}}|_{F_v}$ is a nearly ordinary deformation of $\bar\rho|_{F_v}$, and the restriction of the nearly ordinary character of $\rho^{\mathrm{mod}}_{\mathscr{D}}|_{F_v}$ to I_{F_v} is

$\kappa_{\mathscr{D},v}$ (§7.2, **D2**). The elements $\tilde{U}(p_v)$ and $\tilde{U}(p_v,p_v)$ of $T_{\mathscr{D}}$ are equal to the action of $\rho_{\mathscr{D}}^{\mathrm{mod}}(\sigma)$ on $\mathscr{L}$ and $\det\rho_{\mathscr{D}}^{\mathrm{mod}}\cdot\chi_{\mathrm{cycle}}^{w}(\sigma)$ respectively, where σ is the element of $G_{F_v}^{\mathrm{ab}}$ corresponding to p_v. Thus $\tilde{U}(p_v)$ and $\tilde{U}(p_v,p_v)$ belong to $T_{\mathscr{D}}^{\ss}$.

Next we consider the case when $\mathrm{def}(v)=\mathbf{n.o.f.}$, by assuming that $q_{\bar\rho}$ is one. In this case, we assume that the v-type (k_v,w) is $((2,\dots,2),w)$ (see Definition 6.11), in particular w is even. By twisting by $\chi_{\mathrm{cycle}}^{\frac{w}{2}}$, we may assume that $w=0$. We prepare several lemmas.

Lemma 7.19. *Let D be a division quaternion algebra over F with $q_D=1$ which is split at the places dividing p, K an F-factorizable small compact open subgroup of $G_D(\mathbb{A}_{\mathbb{Q},f})$ such that the v-component K_v is a maximal hyperspecial subgroup of $D^\times(F_v)$ for some place v dividing p.*

(1) *The canonical model $S_{K,F_v}=S_K(G_D(\mathbb{A}_{\mathbb{Q},f}),X_D)_{F_v}$ over F_v has a projective smooth model $S_{K,o_{F_v}}$ over o_{F_v}, which is unique up to canonical isomorphisms. For an inclusion $K'\hookrightarrow K$ of compact open subgroups with $K'_v=K_v$, the induced morphism $S_{K',o_{F_v}}\to S_{K,o_{F_v}}$ is étale.*
(2) *For a discrete infinity type (k,w), assume that the v-type (k_v,w) is $((2,\dots,2),0)$. Then the o_{E_λ}-smooth sheaf $\mathscr{F}_{(k,w)}^{K}$ on S_{K,F_v} admits a unique extension $\tilde{\mathscr{F}}_{(k,w)}^{K}$ as an o_{E_λ}-smooth étale sheaf to $S_{K,o_{F_v}}$.*

Proof of Lemma 7.19. (1) is a consequence of [5]. To be precise, in [5], to discuss integral models over o_{F_v}, compact open subgroups are assumed to be "sufficiently small" outside v, which is stronger than our notion of smallness, though we can reduce to the case stated in the reference. See [15], [25] for the argument.

We show (2). The covering $\pi_p^v:\tilde{S}\to S_{K,F_v}$ which corresponds to $\overline{(F^\times\cap K)}\backslash\prod_{u|p,u\neq v}K_u$ extends to a (pro-) étale coverings of $\tilde{S}_K$ by (1). As in §4.2, $\mathscr{F}_{(k,w)}$ is defined by an o_{E_λ}-lattice of $G_D(\mathbb{Q}_p)$-representation $V_{(k,w),E_\lambda}=\otimes_{\iota:F_u\hookrightarrow E_\lambda,u|p}(\iota\det)^{-k'_\iota}\cdot\mathrm{Sym}^{k_\iota-2}(V_{o_u}\otimes_\iota E_\lambda)^\vee$. By the assumption on the v-type (k_v,w), $V_{(k,w),E_\lambda}$ is regarded as a representation of $\prod_{u|\ell,u\neq v}K_u$. Thus $\mathscr{F}_{(k,w)}$ is defined by the covering π_p^v, and the claim follows.

Lemma 7.20. *Assumptions are as in Lemma 7.19. Let Σ be a set of finite places which contain Σ_K and $\{v:v|p\}$, $T_\Sigma=H(D^\times(\mathbb{A}_{F,f}),K^\Sigma)_{o_{E_\lambda}}$ the convolution algebra over o_{E_λ}, and m_Σ a maximal ideal of T_Σ, which is not of residual type. Then there is a unique p-divisible group E over o_{F_v} such that $T_p(E_{F_v})(-1)$ is isomorphic to $H^1_{\text{ét}}(S_{K,\bar F_v},\ \tilde{\mathscr{F}}_{(k,w),o_{E_\lambda}})_{m_\Sigma}$ as a G_{F_v}-module.*

Proof of Lemma 7.20. Fix $n\geq 1$. By taking a conjugation, we may assume that K_u for $u|p$ is a subgroup of $\mathrm{GL}_2(o_{F_u})$. Let K_n be the compact open subgroup defined by $K_n=K_v\cdot\prod_{u|p,u\neq v}(K(p^n)\cap K_u)\cdot K^p$, and $X_n=S_{K_n,o_{F_v}}$ the integral model for S_{K_n}. For $X=S_{K,o_{F_v}}$, $\pi_n:X_n\to X$ is the covering of X induced by the inclusion $K_n\hookrightarrow K$. $G_n=\overline{F^\times}\cap K_n\backslash K$ acts on X_n, and π_n is a G_n-torsor since K is small.

For $\mathscr{G}_n=\tilde{\mathscr{F}}_{(k,w),o_{E_\lambda}/p^no_{E_\lambda}}^{K}$, by the Hochshild-Serre spectral sequence

$$H^1(G_n,H^0(X_{n,\bar F_v},\pi_n^*\mathscr{G}_n(1)_{\bar F_v}))\longrightarrow H^1(X_{\bar F_v},\mathscr{G}_n(1)_{\bar F_v})\longrightarrow H^1(X_{n,\bar F_v},\pi_n^*\mathscr{G}_n(1)_{\bar F_v})^{G_n}$$
$$\longrightarrow H^2(G_n,H^0(X_{n,\bar F_v},\pi_n^*\mathscr{G}_n(1)_{\bar F_v}))$$

is exact. The action of T_Σ on $H^0(X_{n,\bar F_v},\mathscr{G}_n(1)_{\bar F_v})$ is of residual type by Lemma 5.14. After localization at m_Σ, we have

$$(*)\qquad H^1(X_{\bar F_v},\mathscr{G}_n(1)_{\bar F_v})_{m_\Sigma}\xrightarrow{\sim}(H^1(X_{n,\bar F_v},\pi_n^*\mathscr{G}_n(1)_{\bar F_v})_{m_\Sigma})^{G_n}.$$

By the definition of $\tilde{\mathscr{F}}^K_{(k,w)}$, $\pi_n^*\mathscr{G}_n$ is trivialized on X_n in $D^\times(\mathbb{A}^\Sigma_{F,f})$-equivariant way. Take a trivialization $\pi_n^*\mathscr{G}_n \simeq M_n$, where M_n is a free $o_{E_\lambda}/p^n o_{E_\lambda}$-module.

Let $P_n = Pic^0(X_n/o_{F_v})$ be the connected component of the Picard scheme of X_n over o_{F_v}. We regard the p^n-division points $P_n[p^n]$ of P_n as a finite flat group scheme over o_{F_v}. The convolution algebra T_Σ and G_n acts on $P_n[p^n]$ by the Picard functoriality. We regard $M_n \otimes_{\mathbb{Z}_p/p^n} P_n[p^n]$ as a finite flat group scheme with an action of G_n and T_Σ. G_n acts diagonally on M_n and $P_n[p^n]$, and T_Σ acts trivially on M_n. These two G_n and T_Σ-actions commute. It is easily checked that $H^1(X_{\bar{F}_v}, \mathscr{G}_n(1)_{\bar{F}_v})$ is $G_n \times T_\Sigma$-isomorphic to $M_n \otimes_{\mathbb{Z}_p/p^n} P_n[p^n](\bar{F}_v)$ with the actions defined as above.

The G_n-invariant $(M_n \otimes_{\mathbb{Z}_p/p^n} P_n[p^n])^{G_n}$ is represented by a closed subgroup scheme $\tilde{E}_n$ of $M_n \otimes_{\mathbb{Z}_p/p^n} P_n[p^n]$ ($\tilde{E}_n$ is equal to the Zariski-closure of $(M_n \otimes_{\mathbb{Z}_p/p^n} P_n[p^n])^{G_n}_{F_v}$ in $M_n \otimes_{\mathbb{Z}_p/p^n} P_n[p^n]$). Then we define

$$\mathscr{E}_n = (\tilde{E}_n)_{m_\Sigma}.$$

$\mathscr{E}_n$ is a finite flat group scheme over o_{F_v}. By $(*)$, $\mathscr{E}_n(\bar{F}_v)$ is canonically isomorphic to $H^1(X_{\bar{F}_v}, \mathscr{G}_n(1)_{\bar{F}_v})_{m_\Sigma}$ as a G_{F_v}-module.

For an integer $m \geq 0$, the projection $X_{m+n} \xrightarrow{\pi_{m+n,n}} X_n$ induces $\mathscr{E}_n \to \mathscr{E}_{m+n}$ by the Picard functoriality. By Proposition 5.5, $\{\mathscr{E}_{n,F_v}\}_{n\geq 1}$ forms a p-divisible group over F_v.

Sublemma 7.21. *$\mathscr{E}_n$ is equal to $\mathscr{E}_{m+n}[p^n] = \ker(\mathscr{E}_{m+n} \xrightarrow{p^m} \mathscr{E}_{m+n})$.*

Proof of Sublemma 7.21. $\pi_{m+n,n}$ induces $\pi^*_{m+n,n} : P_n \to P_{m+n}$ by the Picard functoriality. The kernel K of $\pi^*_{m+n,n}$ is finite flat over o_{F_v}. We show that

$$(\dagger) \qquad (K \cap P_n[p^\infty])_{m_\Sigma} = \{0\}.$$

It is sufficient to see $(K \cap P_n[p])_{m_\Sigma} = \{0\}$. $(K \cap P_n[p])(\bar{F}_v)$ is the kernel of $H^1(X_{n,\bar{F}_v}, \mathbb{Z}/p(1)) \to H^1(X_{m+n,\bar{F}_v}, \mathbb{Z}/p(1))$. By Lemma 5.14, the localization at m_Σ is zero, and $(\dagger)$ is shown.

By $(\dagger)$, $P_n[p^n]_{m_\Sigma}$ is regarded as a closed subgroup scheme of P_{m+n}. This implies that $\mathscr{E}_n$ is a closed subgroup scheme of $\mathscr{E}_{m+n}$. Since $\mathscr{E}_n$ and $\mathscr{E}_{m+n}[p^n]$ are two finite flat closed subgroup schemes of $\mathscr{E}_{m+n}$, it is enough to see that the generic fibers over F_v are the same. $\{\mathscr{E}_{n,F_v}\}_{n\geq 1}$ forms a p-divisible group over F_v, so the claim is shown. □

It is clear that the image of $\mathscr{E}_{m+n} \xrightarrow{p^m} \mathscr{E}_{m+n}$ is contained in $\mathscr{E}_n$. Now we have an increasing sequence $\{\mathscr{E}_n\}_{n\geq 1}$ of finite flat group schemes over o_{F_v} which satisfies the following properties:

- For $m, n \geq 1$,
$$0 \longrightarrow \mathscr{E}_n \longrightarrow \mathscr{E}_{m+n} \xrightarrow{p^n} \mathscr{E}_m$$
is exact.
- $\{\mathscr{E}_n\}_{n\geq 1}$ forms a p-divisible group over F_v.

By a celebrated argument of Tate, there is an integer $n_0 \geq 0$ such that $\{\mathscr{E}_{n+n_0}/\mathscr{E}_{n_0}\}_{n\geq 1}$ forms a p-divisible group E over o_{F_v} ([45], p.181–182: the affine algebra of $\mathscr{E}_{n+1}/\mathscr{E}_n$ forms an increasing sequence of orders as n varies, and n_0 is taken so that it becomes stationary for $n \geq n_0$) . By the construction, the Tate module $T_p(E)$ is isomorphic to $H^1(S_{K,\bar{F}_v}, \bar{\mathscr{F}}_{(k,w),o_{E_\lambda}}(1))_{m_\Sigma}$. The uniqueness of E follows from Tate's theorem [45], Theorem 4. □

We go back to the proof of Theorem 7.18. We reduce to the case when the nearly ordinary type $\bar{\kappa}_{\mathscr{D},v}$ is trivial. Take a character $\tau : G_F \to o^\times_{E_\mathscr{D}}$ of finite order which satisfies the following conditions:

- The order of τ is prime to ℓ.
- $\tau|_{I_{F_v}} = \bar{\kappa}_{\mathscr{D},v}$.
- τ is unramified at any finite place u in $(\Sigma_{\mathscr{D}} \setminus \{v\})$.

We can always find such a character τ with these properties by the global class field theory, by replacing $E_{\mathscr{D}}$ by a finite extension if necessary. Let Σ_τ be the ramification set of τ.

Let $\bar{\tau}$ be the mod $m_{\mathscr{D}}$-reduction of τ, $\bar{\rho}' = \bar{\rho} \otimes \bar{\tau}^{-1}$ the twist of $\bar{\rho}$ by $\bar{\tau}^{-1}$. Note that $\bar{\rho}'$ has type 2_{PR} at the places in Σ_τ.

The twist type of $\bar{\rho}'|_{F_u}$ at a finite place u is $\nu_u \otimes \bar{\tau}|_{I_{F_u}}^{-1}$, where ν_u is the twist type of $\bar{\rho}$ at u.

We define a deformation condition $\mathscr{D}'$ for $\bar{\rho}'$. $\mathscr{D}'$ is the same as $\mathscr{D}$ except for the twist types at finite places, and the twist types for $\bar{\rho}'$ is the one defined as above. In particular $\mathrm{def}_{\mathscr{D}} = \mathrm{def}_{\mathscr{D}'}$ holds (at the places in Σ_τ, finite deformations are considered).

It suffices to show the claim for $\bar{\rho}'$ and $\mathscr{D}'$, since the twist of $\rho_{\mathscr{D}}^{\mathrm{mod}}$ by τ^{-1} gives $\rho_{\mathscr{D}'}^{\mathrm{mod}}$ lifting $\bar{\rho}'$.

Thus we may assume that the nearly ordinary type $\bar{\kappa}_{\mathscr{D},v}$ of $\bar{\rho}$ is trivial. We choose an auxiliary place y and a finite set S for $\bar{\rho}$ as in §7.4. Recall that $\tilde{K} = \tilde{K}_{\mathscr{D}_y}(y, S)$ defined in §7.4 is small.

$\tilde{K}_v = \mathrm{GL}_2(o_{F_v})$, and the K-character $\nu_{\mathbf{n.o.f.}}(\bar{\rho}|_{F_v})$ at v is trivial by our assumption.

For $K' = \ker \nu_{\mathscr{D}_y}|_{\tilde{K}}$, let $S_{K',F_v} = S_{K'}(G_{\bar{\rho}}, X_{D(\bar{\rho})})_{F_v}$ be the Shimura curve over F_v. By the definitions of $\tilde{M}_{\mathscr{D}_y}^y$ and $M_{\mathscr{D}}$ (Definition 7.14, 7.16), $M_{\mathscr{D}}$ is a G_F-stable $o_{\mathscr{D}}$-direct summand of $H^1_{\mathrm{stack}}(S_{K'}, \bar{\mathscr{F}}_{(k,w)}^{K'})_{\tilde{m}_{\Sigma,\bar{\rho}}}$. By Lemma 7.20, there is a p-divisible group E' over o_{F_v} such that $T_p(E')(-1)$ is isomorphic to $M_{\mathscr{D}}$ as a G_{F_v}-module. Since $M_{\mathscr{D},E_{\mathscr{D}}}$ is isomorphic to $\rho_{\mathscr{D},E_{\mathscr{D}}}^{\mathrm{mod}}$, by Proposition 3.12, it follows that $\rho_{\mathscr{D}}^{\mathrm{mod}}|_{F_v}$ is nearly ordinary and finite.

Assume that $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{fl}$. We may assume moreover that the v-type is $((2, \ldots, 2), 0)$ and the twist type at v is trivial. As in the case of $\mathbf{n.o.f.}$, there is a p-divisible group E' over o_{F_v} such that $T_p(E')(-1)$ is isomorphic to $M_{\mathscr{D}}$ as a G_{F_v}-module. Since F_v is absolutely unramified and $p \geq 3$, for a G_{F_v}-representation ρ on a finite dimensional $\mathbb{Q}_p$-vector space V and a G_{F_v}-stable lattice L in V, L is associated to a p-divisible group if and only if there is one such lattice L_0 in V. Thus $\rho_{\mathscr{D}}^{\mathrm{mod}}|_{F_v}$ is a flat deformation.

We have shown that $\rho_{\mathscr{D}}^{\mathrm{mod}}|_{F_v}$ is a deformation of type $\mathrm{def}_{\mathscr{D}}(v)$ at any finite place v, and $U(p_v)$, $U(p_v, p_v)$ for $v \in P_{\mathscr{D}}^{\mathbf{u}} \setminus P^{\mathrm{exc}}$, and $\tilde{U}(p_v)$, $\tilde{U}(p_v, p_v)$ for $v \in P^{\mathbf{n.o.}}$ belong to $T_{\mathscr{D}}^{\text{ß}}$. This proves Theorem 7.18 when $q_{\bar{\rho}}$ is one.

7.7. **Approximation method of Taylor.** We prove Theorem 7.18 at ℓ when $q_{\bar{\rho}}$ is zero (under Hypothesis 6.7 if $\mathrm{def}(v) = \mathbf{n.o.}$). We use the approximation method of Taylor [46], [47].

For a finite place u such that $u \notin \Sigma_{\mathscr{D}}$, we assume the following condition:

SP $\bar{\rho}(\mathrm{Fr}_u)$ is a regular semi-simple element of $\mathrm{GL}_2(k)$, and the two distinct eigenvalues of $\bar{\rho}(\mathrm{Fr}_u)$ are of the form $\bar{\alpha}_u$, $q_u\bar{\alpha}_u$.

In the following, we assume that $\bar{\alpha}_u \in k = o_{\mathscr{D}}/m_{\mathscr{D}}$.

$$I_{(k,w)}^{u-\mathrm{par}} = \mathrm{Hom}_{\tilde{K}_{\mathscr{D}} \cap K_0(u)}(\nu_{\mathscr{D}}|_{\tilde{K}_{\mathscr{D}} \cap K_0(u)}, \mathscr{S}_{(k,w)}(G_{\bar{\rho}})).$$

As in §5, $I_{(k,w)}^{K_0(u)}$ contains $I_{(k,w)}^{u-\mathrm{old}} = I_{(k,w),\mathscr{D}}(G_{\bar{\rho}})^{\oplus 2}$ as a subspace (the space of the forms which are old at u). The inclusion is induced by $(f_1,\ f_2) \mapsto \mathrm{pr}_1^* f_1 + \mathrm{pr}_2^* f_2$. Here pr_i $(i = 1, 2)$ are degeneracy maps as in §5.3: pr_1 corresponds to the inclusion $\tilde{K}_{\mathscr{D}} \cap K_0(u) \hookrightarrow \tilde{K}_{\mathscr{D}}$, pr_2

is the projection twisted by the conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & p_u \end{pmatrix}$ at u. The quotient space

$$I_{(k,w)}^{u-\mathrm{new}} = I_{(k,w)}^{u-\mathrm{par}} / I_{(k,w)}^{u-\mathrm{old}}$$

is the space of the forms which are new at u.

Definition 7.22. *Under the condition* **SP** *on u, the Hecke rings $T_{\mathscr{D}}^{u-\mathrm{par}}$ and $T_{\mathscr{D}}^{u-\mathrm{sp}}$ are defined as follows.*

(1) *$\tilde{T}_{\mathscr{D}}^{u-\mathrm{par}}$ is the $o_{\mathscr{D}}$-algebra generated by the following elements in $\mathrm{End}_{o_{\mathscr{D}}} I_{(k,w)}^{u-\mathrm{par}}$: T_v and $T_{v,v}$ for $v \notin \Sigma_{\mathscr{D}} \cup \{u\}$, $U(p_v)$ and $U(p_v, p_v)$ for $v \in P_{\mathscr{D}}^{\mathbf{u}} \setminus P^{\mathrm{exc}}$, $\tilde{U}(p_v)$, and $\tilde{U}(p_v, p_v)$ for $v \in P^{\mathbf{n.o.}}$, $U(p_u)$ and $U(p_u, p_u)$ at u as in Definition 7.2.*

(2) *The maximal ideal $\tilde{m}_{\mathscr{D}}^{u-\mathrm{par}}$ of $\tilde{T}_{\mathscr{D}}^{u-\mathrm{par}}$ is generated by $m_{o_{\mathscr{D}}}$, $T_v - f_{\pi_{\min}}(T_v)$ for $v \notin \Sigma_{\mathscr{D}} \cup \{u\}$, $\tilde{U}(p_v) - \alpha_{\pi_{\min}, v}$ for $v \in P^{\mathbf{n.o.}}$, $U(p_v)$ for $v \in P_{\mathscr{D}}^{\mathbf{u}} \setminus P^{\mathrm{exc}}$, $U(p_u) - \bar{\alpha}_u$, and $U(p_v, p_v) - \chi_{\bar{\rho}}(p_v)$-operators for $v \notin P^{\mathbf{n.o.}} \cup P^{\mathrm{exc}}$ in the notation of 7.2.*

(3) *$T_{\mathscr{D}}^{u-\mathrm{par}}$ is the localization of $\tilde{T}_{\mathscr{D}}^{u-\mathrm{par}}$ at $\tilde{m}_{\mathscr{D}}^{u-\mathrm{par}}$.*

(4) *$\tilde{T}_{\mathscr{D}}^{u-\mathrm{sp}}$ (resp. $\tilde{T}_{\mathscr{D}}^{u-\mathrm{unr}}$) is the image of $\tilde{T}_{\mathscr{D}}^{u-\mathrm{par}}$ in $\mathrm{End}_{o_{\mathscr{D}}} I_{(k,w)}^{u-\mathrm{new}}$ (resp. $\mathrm{End}_{o_{\mathscr{D}}} I_{(k,w)}^{u-\mathrm{old}}$). $\tilde{m}_{\mathscr{D}}^{u-\mathrm{sp}}$ (resp. $\tilde{m}_{\mathscr{D}}^{u-\mathrm{unr}}$) is the image of $\tilde{m}_{\mathscr{D}}^{u-\mathrm{par}}$ in $\tilde{T}_{\mathscr{D}}^{u-\mathrm{sp}}$ (resp. $\tilde{T}_{\mathscr{D}}^{u-\mathrm{unr}}$). Then define $T_{\mathscr{D}}^{u-\mathrm{sp}}$ (resp. $T_{\mathscr{D}}^{u-\mathrm{unr}}$) by $(\tilde{T}_{\mathscr{D}}^{u-\mathrm{sp}})_{\tilde{m}_{\mathscr{D}}^{u-\mathrm{sp}}}$ (resp. $(\tilde{T}_{\mathscr{D}}^{u-\mathrm{unr}})_{\tilde{m}_{\mathscr{D}}^{u-\mathrm{unr}}}$), which are quotients of $T_{\mathscr{D}}^{u-\mathrm{par}}$.*

Lemma 7.23. (1) *$T_{\mathscr{D}}^{u-\mathrm{unr}}$ is isomorphic to $T_{\mathscr{D}}$.*

(2) *We regard $M_{\mathscr{D}}$ as a $T_{\mathscr{D}}^{u-\mathrm{par}}$-module by $T_{\mathscr{D}}^{u-\mathrm{par}} \twoheadrightarrow T_{\mathscr{D}}^{u-\mathrm{unr}} \simeq T_{\mathscr{D}}$. Then $T_{\mathscr{D}}^{u-\mathrm{par}}$-action on $M_{\mathscr{D}}/(U(p_u)^2 - T_{u,u})M_{\mathscr{D}}$ factors through $T_{\mathscr{D}}^{u-\mathrm{sp}}$. In particular $T_{\mathscr{D}}^{u-\mathrm{sp}}$ is non-zero.*

Proof of Lemma 7.23. This is well-known in the theory of congruence modules, so we briefly sketch the proof by explaining how to construct the isomorphisms in Lemma 7.23.

We choose an auxiliary place y and S as in §7.4 such that $u \notin \{y\} \cup S$. By condition **SP**, the polynomial

$$f(X) = X^2 - T_v X + q_v \cdot T_{v,v}$$

has two distinct roots in $T_{\mathscr{D}}^{\ss}/m_{\mathscr{D}}^{\ss}$, and hence in $T_{\mathscr{D}}^{\ss}$ by Hensel's lemma. We choose the root α which lifts $\bar{\alpha}_u$. By the same argument as in the case of **n.o.f.** discussed in §10.3, we obtain an injective homomorphism

$$\xi : M_{\mathscr{D}} \longrightarrow M_{\mathscr{D}}^{u-\mathrm{par}}.$$

Here $K = \ker \nu_{\mathscr{D}_y}|_{\tilde{K}_{\mathscr{D}_y}}$ in the notation of §7.4, and

$$M_{\mathscr{D}}^{u-\mathrm{par}} = \mathrm{Hom}_{\tilde{K}_{\mathscr{D}_y} \cap K_0(v)}(\nu_{\mathscr{D}_y}, H^0_{\mathrm{stack}}(S_{K \cap K_0(v)}, \ \bar{\mathscr{F}}_{(k,w)})_{\tilde{m}_{\Sigma_{\mathscr{D}_y} \cup S \cup \{v\}}}).$$

Since $U(p_u)$ satisfies $f(U(p_u)) = 0$, $U(p_v)$ acts via the multiplication by α on $M_{\mathscr{D}}$.

Since $q_{\bar{\rho}} = 0$, by Theorem 5.10, ξ is universally injective.

$$\xi^{\vee} : M_{\mathscr{D}}^{u-\mathrm{par}} \longrightarrow M_{\mathscr{D}}$$

is the map induced by ξ by Poincaré duality. By Lemma 10.2,

$$\xi^{\vee} \circ \xi(M_{\mathscr{D}}) = (U(p_u)^2 - T_{u,u})M_{\mathscr{D}}$$

as in the case of **n.o.f.** in §10.2. By a standard argument in the theory of congruence modules [38], the congruence module $M_{\mathscr{D}}/\xi^{\vee} \circ \xi(M_{\mathscr{D}})$ is a $T_{\mathscr{D}}^{u-\mathrm{par}}$-module, and Lemma 7.23 is shown. $\square$

By $T_{\mathscr{D}}^{\text{ß},u-\text{par}}$ we denote the $o_{\mathscr{D}}$-subalgebra of $T_{\mathscr{D}}^{u-\text{par}}$ generated by $T_v,\ v \notin \Sigma_{\mathscr{D}} \cup \{u,y\} \cup S$ as in 7.2, and denote by $T_{\mathscr{D}}^{\text{ß},u-\text{sp}}$ (resp. $T_{\mathscr{D}}^{\text{ß},u-\text{unr}}$) the image of $T_{\mathscr{D}}^{\text{ß},u-\text{par}}$ in $T_{\mathscr{D}}^{u-\text{sp}}$ (resp. $T_{\mathscr{D}}^{u-\text{unr}}$). By Lemma 7.8, $T_{\mathscr{D}}^{\text{ß},u-\text{unr}} = T_{\mathscr{D}}^{\text{ß}}$.

Lemma 7.24. *For any integer $N \geq 1$, there is a finite place u which satisfies the following properties:*

(1) *$u \notin \Sigma_{\mathscr{D}}$, $q_u \equiv -1 \mod \ell$, and the eigenvalues of $\bar{\rho}(\mathrm{Fr}_u)$ are of the form $\bar{\alpha},\ -\bar{\alpha}$. In particular* **SP** *holds at u.*

(2) *We define $T_{\mathscr{D}}^{u-\text{par}}$ and $T_{\mathscr{D}}^{u-\text{sp}}$ by choosing an eigenvalue of $\bar{\rho}(\mathrm{Fr}_u)$. Then $T_{\mathscr{D}}^{\text{ß}}/m_{\mathscr{D}}^N T_{\mathscr{D}}^{\text{ß}} = T_{\mathscr{D}}^{\text{ß},u-\text{unr}}/m_{\mathscr{D}}^N T_{\mathscr{D}}^{\text{ß},u-\text{unr}}$ is a quotient of $T_{\mathscr{D}}^{\text{ß},u-\text{sp}}$ as a $T_{\mathscr{D}}^{\text{ß},u-\text{par}}$-algebra.*

Proof of Lemma 7.24. By taking a finite unramified extension of $o_{\mathscr{D}}$, we may assume that all eigenvalues of $\bar{\rho}(g)$, $g \in G_F$ belong to $k = o_{\mathscr{D}}/m_{\mathscr{D}}$.

$M_{\mathscr{D},E_{\mathscr{D}}}$ is free of rank one as a $T_{\mathscr{D},E_{\mathscr{D}}}^{\text{ß}}$-module by Proposition 7.15, Lemma 7.6, and Lemma 7.13. We choose an injective $T_{\mathscr{D}}^{\text{ß}}$-homomorphism with a finite cokernel

$$i : T_{\mathscr{D}}^{\text{ß}} \hookrightarrow M_{\mathscr{D}},$$

and take $c_0 \geq 0$ such that $m_{\mathscr{D}}^{c_0}$ annihilates $\operatorname{coker} i$.

Let $\tilde{T}$ be the normalization of $T_{\mathscr{D}}^{\text{ß}}$. Since $T_{\mathscr{D}}^{\text{ß}}$ is reduced, the canonical $o_{\mathscr{D}}$-algebra homomorphism gives an embedding

$$j : T_{\mathscr{D}}^{\text{ß}} \hookrightarrow \tilde{T}.$$

$c_1 \geq 0$ is an integer such that $m_{\mathscr{D}}^{c_1}$ annihilates $\operatorname{coker} j$.

Define $\rho_{\tilde{T}} : G_F \to \mathrm{GL}_2(\tilde{T})$ as the composition of $G_F \overset{\rho_{\mathscr{D}}^{\text{mod}}}{\to} \mathrm{GL}_2(T_{\mathscr{D}}^{\text{ß}}) \to \mathrm{GL}_2(\tilde{T})$.

We fix an element c of order 2 in G_F which corresponds to the complex conjugation for some embedding $F \hookrightarrow \mathbb{C}$. We view $\mathscr{L} = \tilde{T}^{\oplus 2}$ as a G_F-module by $\rho_{\tilde{T}}$. For $\gamma = c_1 + c_2$, let H_N be the image of G_F in $\mathrm{Aut}((\mathbb{Z}/\ell^{N+\gamma}\mathbb{Z})(1) \oplus \mathscr{L}/m_{\mathscr{D}}^{N+\gamma}\mathscr{L})$, C the conjugacy class of c in H_N. We take u so that Fr_u belongs to C by the Chebotarev density theorem. We show that u satisfies the desired properties.

By our choice of u, 7.24 (1) is satisfied, $1 + q_u \equiv 0 \mod \ell^{N+\gamma}$ holds, and the image of T_u is zero in $\tilde{T}/m_{\mathscr{D}}^{N+\gamma}\tilde{T}$.

Since $\operatorname{coker} j$ is annihilated by $m_{\mathscr{D}}^{c_1}$, T_u is contained in $m_{\mathscr{D}}^{N+\gamma-c_1}T_{\mathscr{D}}^{\text{ß}} = m_{\mathscr{D}}^{N+c_0}T_{\mathscr{D}}^{\text{ß}}$.

We choose an eigenvalue $\bar{\alpha}$ of $\bar{\rho}(\mathrm{Fr}_u)$, and define Hecke algebras $T_{\mathscr{D}}^{u-\text{par}}$ and $T_{\mathscr{D}}^{u-\text{sp}}$ using $\bar{\alpha}$. Let α be the roots of $f(X) = X^2 - T_u X + q_u \cdot T_{u,u}$ which lift $\bar{\alpha}$. The action of $U(p_u)$ on $M_{\mathscr{D}}$ is the multiplication by α in $T_{\mathscr{D}}^{\text{ß}}$.

As

$$A_u = \alpha^2 - T_{u,u} = T_u \cdot \alpha - (1 + q_u)T_{u,u} \in m_{\mathscr{D}}^{N+c_0}T_{\mathscr{D}}^{\text{ß}},$$

$M_{\mathscr{D}}/(U(p_u)^2 - T_{u,u})M_{\mathscr{D}} = M_{\mathscr{D}}/A_u M_{\mathscr{D}}$ admits $M_{\mathscr{D}}/m_{\mathscr{D}}^{N+c_0}M_{\mathscr{D}}$ as a quotient. By Lemma 7.23 (2), $M_{\mathscr{D}}/m_{\mathscr{D}}^{N+c_0}M_{\mathscr{D}}$ is a $T_{\mathscr{D}}^{\text{ß},u-\text{sp}}$-module.

By our choice of c_0, the image of $i(T_{\mathscr{D}}^{\text{ß}})$ in $M_{\mathscr{D}}/m_{\mathscr{D}}^{N+c_0}M_{\mathscr{D}}$ admits $T_{\mathscr{D}}^{\text{ß}}/m_{\mathscr{D}}^N T_{\mathscr{D}}^{\text{ß}}$ as a quotient. Thus 7.24, (2) is shown.

□

For an integer $N \geq 1$, we choose a finite place u_N which fulfills the conclusion of Lemma 7.24.

We have a surjective $o_{\mathscr{D}}$-algebra homomorphism

$$\alpha_N : T_{\mathscr{D}}^{\text{ß},u_N-\text{sp}} \twoheadrightarrow T_{\mathscr{D}}^{\text{ß}}/m_{\mathscr{D}}^N T_{\mathscr{D}}^{\text{ß}}$$

which maps T_v to T_v for $v \notin \Sigma_{\mathscr{D}} \cup \{u_N\}$.

For $T_{\mathscr{D}}^{\text{ß},u_N-\mathrm{sp}}$, there is a Galois representation $\rho^{\mathrm{mod}}_{\mathscr{D},u_N-\mathrm{sp}} : G_F \to \mathrm{GL}_2(T_{\mathscr{D}}^{\text{ß},u_N-\mathrm{sp}})$ as in Proposition 7.10. Since any representation π which appear as a component of $T_{\mathscr{D}}^{\text{ß},u_n-\mathrm{sp}}$ has a special representation π_{u_N} at u_N, we can apply Hypothesis 6.7 to π, and show that $\rho^{\mathrm{mod}}_{\mathscr{D},u_N-\mathrm{sp}}$ is nearly ordinary, nearly ordinary finite or flat according to $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{n.o.}$, $\mathbf{n.o.f.}$, $\mathbf{fl}$ by the argument in §7.6.

By the surjectivity of α_N and the uniqueness (Lemma 7.8), $\rho^{\mathrm{mod}}_{\mathscr{D}} \mod m^n_{\mathscr{D}} = \alpha_N \circ \rho^{\mathrm{mod}}_{\mathscr{D},u_N-\mathrm{sp}}$ is nearly ordinary or nearly ordinary finite at a place v such that $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{n.o.}$, $\mathbf{n.o.f.}$, $\mathbf{fl}$. Since this is true for any N, the claim for $T_{\mathscr{D}}$ is shown. □

Remark 7.25. *In the* $\mathbf{fl}$*-case, one may use Taylor's result (*[47]*, theorem 1.6 and lemma 2.1, 3) if the infinity type is* $((2,\dots,2),w)$*. The demonstration there can be modified so that it fits to our setting (we only assume that the* v*-type is* $((2,\dots,2),w)$*).*

Proposition 7.26. *The following operators in* $\mathrm{End}_{o_{\mathscr{D}}} M_{\mathscr{D}}$ *belong to* $T^{\text{ß}}_{\mathscr{D}}$*.* $U(p_v), U(p_v,p_v)$ *for* $v \in P^{\mathbf{f}}_{\mathscr{D}}$*,* $\tilde{U}(p_v), \tilde{U}(p_v,p_v)$ *for* $v \in P^{\mathbf{n.o.f.}} \cup P^{\mathbf{fl}}$*.*

Proof of Proposition 7.26. By Lemma 7.3, the operators in question belong to $T^{\text{ß}}_{\mathscr{D},E_{\mathscr{D}}}$. We find these operators by using $\rho^{\mathrm{mod}}_{\mathscr{D}}$. For a finite place v, let σ_v be an element in I_{F_v} which lifts the geometric Frobenius element, and corresponds to p_v by $(I^{\mathrm{ab}}_{F_v})_{G_{F_v}} \simeq o^{\times}_{F_v}$.

For $v \in P^{\mathbf{f}}_{\mathscr{D}}$, we use the notation of §7.5. $U(p_v,p_v) = q_v^{-1}\det\rho^{\mathrm{mod}}_{\mathscr{D}}(\sigma_v)$. When $d_v = 2$, $U(p_v) = \mathrm{trace}_{T^{\text{ß}}_{\mathscr{D}}}\rho(\sigma_v)$. When $d_v = 1$, $U(p_v)$ is equal to the action of $\rho^{\mathrm{mod}}_{\mathscr{D}}(\sigma_v)$ on $\mathscr{L}$. Note that it is sufficient to verify the equalities after tensoring $E_{\mathscr{D}}$.

For $v \in P^{\mathbf{n.o.f.}} \cup P^{\mathbf{fl}}$, $\tilde{U}(p_v,p_v) = (\det\rho^{\mathrm{mod}}_{\mathscr{D}} \cdot \chi^{w+1}_{\mathrm{cycle}})(p_v)$ holds. For $\tilde{U}(p_v)$-operator, by twisting, we may assume that $w = 0$. By the argument (twisting by a finite order character) in §7.6, we may assume that the twist type at v is trivial.

For $\rho_v = \rho^{\mathrm{mod}}_{\mathscr{D}}|_{G_{F_v}}$, $\rho_v(1)$ extends to a unique ℓ-divisible group E over o_{F_v}, and it is a crystalline representation.

If $v \in P^{\mathbf{n.o.f.}}$, let $\tilde{U}$ be the action of $\rho_v(\sigma_v)$ on $\mathscr{L}$ in the notation of §7.6. We define $\tilde{T}_v$ by

$$\tilde{T}_v = \tilde{U} + q_v\tilde{U}(p_v,p_v)\cdot\tilde{U}^{-1}.$$

If $v \in P^{\mathbf{fl}}$, let $D(\rho_v)$ be the filtered module associated to ρ_v of filtration type $[0,\ 1]$ by [14]. $D(\rho_v)$ is a free $T^{\text{ß}}_{\mathscr{D}} \otimes_{\mathbb{Z}_\ell} o_{F_v}$-module with an action of the absolute Frobenius φ. We define $\tilde{T}_v$ by

$$\tilde{T}_v = \mathrm{trace}_{T^{\text{ß}}_{\mathscr{D}}\otimes_{\mathbb{Z}_\ell} o_{F_v}}(\varphi^f,\ D(\rho_v)) \in T^{\text{ß}}_{\mathscr{D}} \otimes_{\mathbb{Z}_\ell} o_{F_v}.$$

Here $f = [k(v) : \mathbb{F}_\ell]$. φ^f is $T^{\text{ß}}_{\mathscr{D}} \otimes_{\mathbb{Z}_\ell} o_{F_v}$-linear, thus $\tilde{T}_v$ is well-defined.

We show

$$\tilde{U}(p_v) = \tilde{T}_v$$

in $T^{\text{ß}}_{\mathscr{D}} \otimes_{\mathbb{Z}_\ell} o_{F_v}$ in the both cases. It suffices to see it in $T^{\text{ß}}_{\mathscr{D}} \otimes_{\mathbb{Z}_\ell} F_v$. If $q_{\bar\rho} = 1$, this is a consequence of the main theorem of [42], which shows a weaker version of the compatibility of the local and the global Langlands correspondence at $v|\ell$.

When $q_{\bar\rho}$ is zero, we use the approximation method using Lemma 7.24, and the equality holds in $T^{\text{ß}}_{\mathscr{D}} \otimes_{\mathbb{Z}_\ell} o_{F_v}/m^N_{F_v}$ for any N, and hence in $T^{\text{ß}}_{\mathscr{D}} \otimes_{\mathbb{Z}_\ell} o_{F_v}$.

Since o_{F_v} is faithfully flat over $\mathbb{Z}_\ell$, $\tilde{U}(p_v) \in T^{\text{ß}}_{\mathscr{D}}$. □

8. Construction of Taylor-Wiles systems

In this section, we construct the Taylor-Wiles system for a minimal Hecke algebra $T_{\mathscr{D}}$.

Theorem 8.1. *Let $\ell \geq 3$ be a prime, $\bar\rho$ an absolutely irreducible representation. Assume that $\bar\rho|_{F(\zeta_\ell)}$ is absolutely irreducible if $[F(\zeta_\ell):F]=2$.*

For a minimal deformation type $\mathscr{D}$, define $o_{\mathscr{D}}^{\mathrm{ass}}$ as $o_{\mathscr{D}}[C_{F,\ell}]$, where $C_{F,\ell}$ is the ℓ-part of the class group of F.

Then there is a Taylor-Wiles system $\{R_Q,\ M_Q\}_{Q\in\mathscr{X}_{\mathscr{D}}}$ for $(R_{\mathscr{D}}, M_{\mathscr{D}})$ and torus $\mathbb{G}_{m,F}$ over $o_{\mathscr{D}}^{\mathrm{ass}}$. The index set $\mathscr{X}_{\mathscr{D}}$ is defined by

$$Q\in\mathscr{X}_{\mathscr{D}} \Longleftrightarrow Q \text{ is a finite subset of } Y_{\mathscr{D}}.$$

Here, $Y_{\mathscr{D}}$ is the set of the finite places $v\notin\Sigma_{\mathscr{D}}$, $q_v\equiv 1 \mod \ell$, and $\bar\rho(\mathrm{Fr}_v)$ is a regular semi-simple element of $\mathrm{GL}_2(k)$.

In [49], TW5 was obtained by a study of group cohomology. We use an argument based on a property of perfect complexes.

8.1. **Hecke algebras.** For an absolutely irreducible representation $\bar\rho$, we fix a deformation type $\mathscr{D}$ for $\bar\rho$.

For an element $Q\in\mathscr{X}_{\mathscr{D}}$, we construct a Hecke algebra T_Q and a T_Q-module M_Q, define a ring homomorphism $R_Q\to T_Q$ from some deformation ring R_Q, and verify the conditions of Taylor-Wiles systems.

By taking an extension of k, we may assume that any eigenvalues of $\bar\rho(g)$ for $g\in G_F$ belong to k. For a finite place $v\in Y_{\mathscr{D}}$, let $\bar\alpha_v$ and $\bar\beta_v$ be the eigenvalues of $\bar\rho(\mathrm{Fr}_v)$ at v. We make a choice, and specify one of the eigenvalues of $\bar\rho(\mathrm{Fr}_v)$, say $\bar\alpha_v$. We denote by Δ_v (resp. $\Delta_v^\vee$) the ℓ-Sylow subgroup of $k(v)^\times$ (resp. the maximal prime to ℓ-subgroup of $k(v)^\times$).

For $Q\in\mathscr{X}_{\mathscr{D}}$, we choose an auxiliary place y and a set S as in §7.4 which are disjoint from $\Sigma_{\mathscr{D}}\cup Q$. We define the deformation function $\mathrm{def}_{\mathscr{D}_Q}$ of $\bar\rho$ by

$$\mathrm{def}_{\mathscr{D}_Q}(v)=\mathrm{def}_{\mathscr{D}}(v) \text{ for } v\notin\Sigma_{\mathscr{D}},\quad \mathbf{u} \text{ for } v\in Q,$$

and let $\mathscr{D}_Q$ be the deformation type of $\bar\rho$ which has $\mathrm{def}_{\mathscr{D}_Q}$ as the deformation function, and the same data as $\mathscr{D}$ except for the deformation function.

Let $K=\tilde K_{\mathscr{D}_y}=K_{11}(m_y^2)\cdot K_{\mathscr{D}}^y$ be the compact open subgroup of $G_{\bar\rho}(\mathbb{A}_{\mathbb{Q},f})$ defined in §7.4 for $\mathscr{D}_y$. We define two compact open subgroups $K_{0,Q}$, K_Q of K for Q:

$$K_{0,Q}=\prod_{v\in Q}K_0(m_v)\cdot K^Q,$$

$$K_Q=\prod_{v\in Q}K_{Q,v}\cdot K^Q.$$

Here

$$K_{Q,v}=\{g\in K_0(m_v),\quad g\equiv\begin{pmatrix}\alpha & *\\ 0 & \alpha\cdot h\end{pmatrix} \mod m_v,\ \alpha\in k(v)^\times,\ h,\in\Delta_v^\vee\}.$$

There are inclusions $K_Q\subset K_{0,Q}\subset K$. K_Q is a normal subgroup in $K_{0,Q}$, and the quotient $(\overline{Z(\mathbb{Q})}\cap K_Q)\cdot K_Q\backslash K_{0,Q}$ is isomorphic to

$$\Delta_Q:=\prod_{v\in Q}\Delta_v.$$

We define the Hecke algebras for $K_{0,Q}$ and K_Q which are slightly different from the one defined for $\mathscr{D}$ and $\mathscr{D}_Q$ in §7 at the places in Q.

For $\delta\in\Delta_Q$, we take an element $\delta'\in\prod_{v\in Q}o_{F_v}^\times$ lifting δ and $c(\delta')\in G_{\bar\rho}(\mathbb{A}_{\mathbb{Q},f})$ is the element which has $c(\delta')_Q=\begin{pmatrix}\delta'^{-1} & 0\\ 0 & \delta'\end{pmatrix}$, at the Q-component and the components other

than Q are one. Then the double coset $K_Q c(\delta') K_Q$ depends only on δ, and defines an element $V(\delta)$ in $H(G_D(\mathbb{A}_{\mathbb{Q},f}),\ K_Q)$.

By our assumption, there is a minimal lift $\pi_{\min}$ of $\bar{\rho}$ defined over $o_{\mathscr{D}}$. $f_{\pi_{\min}} : T^{\text{ß}}_{K,\mathscr{D}} \to o_{\mathscr{D}}$ is the corresponding $o_{\mathscr{D}}$-algebra homomorphism.

Definition 8.2. (1) *$I^0_{(k,w),\mathscr{D}_Q}$ (resp. $I_{(k,w),\mathscr{D}_Q}$) is the intertwining space*

$$I^0_{(k,w),\mathscr{D}_Q} = \mathrm{Hom}_{K_{0,Q}}(\nu_{\mathscr{D}}|_{K_{0,Q}}, \mathscr{S}_{(k,w)}(G_{\bar{\rho}})) \quad (\textit{resp. } I_{(k,w),\mathscr{D}_Q} = \mathrm{Hom}_{K_Q}(\nu_{\mathscr{D}}|_{K_Q}, \mathscr{S}_{(k,w)}(G_{\bar{\rho}})).$$

(2) *For an element $\pi \in \mathscr{A}_{(k,w)}(G_{\bar{\rho}})$, $I^0_{(k,w),\mathscr{D}_Q}(\pi)$ (resp. $I_{(k,w),\mathscr{D}_Q}(\pi)$) is the intertwining space*

$$I^0_{(k,w),\mathscr{D}_Q}(\pi) = \mathrm{Hom}_{K_{0,Q}}(\nu_{\mathscr{D}}|_{K_{0,Q}}, \pi_f) \quad (\textit{resp. } I_{(k,w),\mathscr{D}_Q} = \mathrm{Hom}_{K_Q}(\nu_{\mathscr{D}}|_{K_Q}, \pi_f)).$$

Definition 8.3. (1) *The Hecke algebra $\tilde{T}_Q$ is the $o_{\mathscr{D}}$-algebra generated by the following elements in $\mathrm{End}_{o_{\mathscr{D}}} I_{(k,w),\mathscr{D}_Q}$: T_v, and $T_{v,v}$ for $v \notin \{y\} \cup \Sigma_{\mathscr{D}_Q}$, $U(p_v)$ and $U(p_v,p_v)$ for $v \in \{y\} \cup S \cup P^{\mathbf{u}}_{\mathscr{D}_Q} \setminus P^{\mathrm{exc}}$, $\tilde{U}(p_v)$ and $\tilde{U}(p_v,p_v)$ for $v \in P^{\mathbf{n.o.}}$, $V(\delta)$ for $\delta \in \Delta_Q$.*

$\tilde{T}_{0,Q}$ is the image of $\tilde{T}_Q$ in $\mathrm{End}_{o_{\mathscr{D}}} I^0_{(k,w),\mathscr{D}_Q}$.

(2) *By $\tilde{m}_Q$, we denote the maximal ideal of $\tilde{T}_Q$ generated by $m_{o_{\mathscr{D}}}$ and the following operators: $T_v - f_{\pi_{\min}}(T_v)$, $T_{v,v} - f_{\pi_{\min}}(T_{v,v})$ for $v \notin \{y\} \cup S \cup \Sigma_{\mathscr{D}_Q}$, $U(p_v)$, $U(p_v,p_v) - \chi_{\bar{\rho}}(p_v)$ for $v \in (\{y\} \cup P^{\mathbf{u}}_{\mathscr{D}}) \setminus P^{\mathrm{exc}}$, $\tilde{U}(p_v) - \alpha_{\pi_{\min},v}$, $\tilde{U}(p_v,p_v) - \gamma_{\pi_{\min}}$ for $v \in P^{\mathbf{n.o.}}$, $U(p_v) - \bar{\alpha}_v$, $U(p_v,p_v) - \chi_{\bar{\rho}}(p_v)$ for $v \in Q$, $V(\delta) - 1$ for $\delta \in \Delta_Q$. Here $\alpha_{\pi_{\min},v}$ and $\gamma_{\pi_{\min}}$ are elements of $o_{\mathscr{D}}$ defined in §7.2.*

$\tilde{m}_{0,Q}$ is the image of $\tilde{m}_Q$ in $\tilde{T}_{0,Q}$.

(3) *The Hecke algebras $T_{0,Q}$ and T_Q associated to Q are defined by*

$$T_{0,Q} = (\tilde{T}_{0,Q})_{\tilde{m}_{0,Q}}, \quad T_Q = (\tilde{T}_Q)_{\tilde{m}_Q}.$$

Note that the natural $o_{\mathscr{D}}$-algebra homomorphism $T_Q \twoheadrightarrow T_{0,Q}$ maps $V(\delta)$ for $\delta \in \Delta_Q$ to 1.

As in §7.2, we define $T^{\text{ß}}_{0,Q}$ (resp. $T^{\text{ß}}_Q$) as the $o_{\mathscr{D}}$-subalgebra generated by T_v and $T_{v,v}$ for $v \notin \{y\} \cup S \cup \Sigma_{\mathscr{D}_Q}$.

Proposition 8.4. *$T^{\text{ß}}_{0,Q} = T_{0,Q}$ and $T^{\text{ß}}_Q = T_Q$ hold. In particular $T_{0,Q}$, T_Q are reduced.*

Proof of Proposition 8.4. We proceed as in the proofs of Lemma 7.6 and Proposition 7.12. First we show $T^{\text{ß}}_{Q,E_{\mathscr{D}}} = T_{Q,E_{\mathscr{D}}}$.

Let $e_{\bar{\rho}}$ be the idempotent of $\tilde{T}_Q$ corresponding to T_Q. Take a representation $\pi \in \mathscr{A}_{(k,w)}(G_{\bar{\rho}})$ such that $e_{\bar{\rho}} I_{(k,w),\mathscr{D}_Q}(\pi) \neq \{0\}$. We show the dimension over $\bar{E}_{\mathscr{D}}$ is one. Since $I_{(k,w),\mathscr{D}_Q}(\pi)$ is the tensor product local spaces, it suffices to study the local intertwining space at any finite place. If $v \notin \{y\} \cup Q$, this is similarly treated as in Lemma 7.6. At y, this is done in Proposition 7.12. It suffices to consider the case of $v \in Q$.

If $v \in Q$, the v-compenent π_v belongs to principal series or a special representation twisted by a character since it has a non-zero fixed vector by $K_{11}(m_v)$. Since we assume that $q_v \equiv 1 \mod \ell$ and two eigenvalues $\bar{\alpha}_v$, $\bar{\beta}_v$ of $\bar{\rho}(\mathrm{Fr}_v)$ are distinct modulo $m_{\mathscr{D}}$, the latter case does not occur. Since π_v has a non-zero fixed vector by K_v, π_v is a tamely ramified principal series.

Lemma 8.5. *For a local field F and a principal series representation $\pi = \pi(\chi_1, \chi_2)$, assume that χ_1 and χ_2 are tamely ramified characters of $F^\times$. Then there is a unique non-zero vector w up to scalar in the representation space such that*

$$\begin{pmatrix} a & * \\ p_F \cdot * & d \end{pmatrix} \cdot w = \chi_1(a)\chi_2(d) w \quad (a,\ d \in o_F^\times).$$

Proof of Lemma 8.5. Consider $\pi' = \pi \otimes \chi_2^{-1} = \pi(\chi_1/\chi_2, 1)$. A new vector w' fixed by $K_1(m_F)$ of π' corresponds to w as in the claim by twisting by χ_2^{-1}. □

We choose an ordering of χ_1 and χ_2 so that $\chi_2(p_v)$ is a lift of $\bar{\alpha}_v$. Since $\bar{\alpha}_v \neq \bar{\beta}_v$, this determines the ordering uniquely. By Sublemma 8.5 and Lemma 3.30, $I_{\mathbf{u}}(\bar{\rho}|_{F_v}, \pi_v) = \pi_v^{K_{11}(m_v)}$ is isomorphic to $\bar{E}_{\mathscr{D}}[U]/((U-\chi_1(p_v))(U-\chi_2(p_v)))$, where the action of U is $U(p_v)$. It becomes one dimensional after localization at $(U-\chi_2(p_v))$. Since the local intertwining spaces have dimension one, it follows that $e_{\bar{\rho}} I_{(k,w),\mathscr{D}_Q}(\pi)$ is one dimensional.

This proves that $T^{ß}_{Q,E_{\mathscr{D}}} = T_{Q,E_{\mathscr{D}}}$ as in Lemma 7.6. In particular T_Q is reduced. From this, we construct Galois representation

$$\rho_Q^{\mathrm{mod}} : G_{\Sigma_{\mathscr{D}_Q}} \longrightarrow \mathrm{GL}_2(T_Q^{ß})$$

as in Proposition 7.10. We show $T_Q^{ß} = T_Q$ by an argument using Galois representations as in Theorem 7.18. The only point we must discuss is to recover $V(\delta)$-operators for $\delta \in \Delta_Q$ from ρ_Q^{mod}.

Lemma 8.6. (1) *For $v \in Q$, $\rho_Q^{\mathrm{mod}}|_{G_{F_v}}$ is a sum of two characters $\chi_{1,v}$, $\chi_{2,v} : G_{F_v} \to T_Q^{\times}$.*

(2) *Fix the ordering of the characters such that $\bar{\chi}_{2,v}(\mathrm{Fr}_v) = \bar{\alpha}_v$. For an element σ_δ of $I_{F_v}^{\mathrm{tame}}$ which lifts $\delta \in \Delta_v$ via $I_{F_v}^{\mathrm{tame}} \to (I_{F_v}^{\mathrm{tame}})_{G_F} \simeq k(v)^{\times}$, $\chi_{2,v}(\sigma_\delta) = V(\delta)$.*

Proof of Lemma 8.6. (1) follows from Faltings' theorem 3.8.

For (2), take a representation $\pi \in \mathscr{A}_{(k,w)}(G_{\bar{\rho}})$ such that $e_{\bar{\rho}} I_{(k,w),\mathscr{D}_Q}(\pi) \neq \{0\}$. It suffices to check the identity in (2) on the component of T_Q corresponding to π. For a finite extension E_λ of $E_{\mathscr{D}}$ such that π_f is defined over E_λ, let $f_\pi : T_Q^{ß} \to o_{E_\lambda}$ be the $o_{\mathscr{D}}$-homomorphism corresponding to π. f_π induces $\rho : G_F \to \mathrm{GL}_2(T_Q^{ß}) \to \mathrm{GL}_2(o_{E_\lambda})$, which is isomorphic to the λ-adic representation associated to π. By the compatibility of the local and the global Langlands correspondence, π_v corresponds to $\rho|_{G_{F_v}}$ by the local Langlands correspondence. $\rho|_{G_{F_v}} \simeq \tilde{\chi}_{1,v} \oplus \tilde{\chi}_{2,v}$, where $\chi_{i,v}$ is the composition of $G_{F_v} \xrightarrow{\chi_{i,v}} T_Q^{\times} \xrightarrow{f_\pi} o_{E_\lambda}^{\times}$. It follows that π_v is isomorphic to the principal series $\pi(\tilde{\chi}_1, \tilde{\chi}_2)$. By Lemma 8.5, the action of $V(\delta)$ on $\pi_v^{K_{11}(m_v)}$ is $\tilde{\chi}_2(\sigma_\delta)$, and the claim follows. □

By Lemma 8.6, $V(\delta)$ belongs to $T_Q^{ß}$. We conclude that $T_Q^{ß} = T_Q$, and $T_{0,Q}^{ß} = T_{0,Q}$ follows from this as a consequence. □

Let $T_{\mathscr{D}_y}^{ß}$ and $T_{\mathscr{D}_y}$ be the ℓ-adic Hecke algebras defined in Definition 7.2. $T_{\mathscr{D}_y}^{ß} = T_{\mathscr{D}_y}$, and $T_{\mathscr{D}_y} \simeq T_{\mathscr{D}}$ by Theorem 7.18 and Proposition 7.12.

For ℓ-adic reduced Hecke algebras we have defined, there are $o_{\mathscr{D}}$-algebra homomorphisms $T_Q^{ß} \to T_{0,Q}^{ß} \to T_{\mathscr{D}_y}^{ß}$. By the definition and Lemma 7.8, these homomorphisms are surjective, and induce surjective homomorphisms between ℓ-adic Hecke algebras

$$T_Q \twoheadrightarrow T_{0,Q} \twoheadrightarrow T_{\mathscr{D}_y}.$$

by Proposition 8.4.

Proposition 8.7. *The above $o_{\mathscr{D}}$-algebra homomorphism gives an isomorphism*

$$T_{0,Q} \simeq T_{\mathscr{D}_y}.$$

Proof of Proposition 8.7. First we show $T_{0,Q,E_{\mathscr{D}}} \simeq T_{\mathscr{D}_y,E_{\mathscr{D}}}$.

Let $e_{0,\bar{\rho}}$ be the idempotent of $\tilde{T}_{0,Q}$ corresponding to $T_{0,Q}$. Take a representation $\pi \in \mathscr{A}_{(k,w)}(G_{\bar{\rho}})$ such that $e_{\bar{\rho}}^0 I_{(k,w),\mathscr{D}_Q}(\pi) \neq \{0\}$. We show π_v is spherical for $v \in Q$.

By our choice of $K_{0,Q}$, the central character of π_v is unramified. If π_v is not spherical, it is an unramified special representation since it is fixed by $K_0(m_v)$. Let $\rho_{\pi,\bar{E}_{\mathscr{D}}}$ be the Galois representation associated to π. By the compatibility of the local and the global Langlands correspondence, $\rho_v = \rho_{\pi,\bar{E}_{\mathscr{D}}}|_{G_{F_v}} \simeq \mathrm{sp}(2) \otimes \chi_v$, where $\mathrm{sp}(2)$ is the special representation, and χ_v is an unramified character. This implies that for any Frobenius lift $\sigma \in G_{F_v}$, the two eigenvalues α_v and β_v of $\rho_v(\sigma)$ must satisfy $\alpha_v/\beta_v = q_v^{\pm 1}$. Since $q_v \equiv 1 \mod \ell$ and $\bar{\alpha}_v$, $\bar{\beta}_v$ are distinct, this is a contradiction. Thus π_v is spherical at $v \in Q$, and π appear as a component of $T_{\mathscr{D}_y}$.

As in the proof of Proposition 8.4, $e^0_{\bar{\rho}} I^0_{(k,w),\mathscr{D}_Q}(\pi)$ is one dimensional over $\bar{E}_{\mathscr{D}}$, and the action of $T_v \in T_{0,Q}$ for $v \notin \{y\} \cup S \cup \Sigma_{\mathscr{D}_Q}$ is the same as the action of $T_v \in T_{\mathscr{D}_y}$ on $e_{\bar{\rho}} I^y_{(k,w),\mathscr{D}_y}(\pi)$ in the notation of §7.4. Thus we have an isomorphism $T_{0,Q,E_{\mathscr{D}}} \simeq T_{\mathscr{D}_y,E_{\mathscr{D}}}$, which induces $T^{\text{ß}}_{0,Q} \simeq T^{\text{ß}}_{\mathscr{D}_y}$.

To show $T_{0,Q,E_{\mathscr{D}}} \simeq T_{\mathscr{D}_y,E_{\mathscr{D}}}$, it suffices to see $U(p_v)$ and $U(p_v,p_v)$ belong to $T^{\text{ß}}_{0,Q}$. For $v \in Q$, note that there are elements T_v and $T_{v,v}$ in $T^{\text{ß}}_{0,Q} \simeq T_{\mathscr{D}_y,E_{\mathscr{D}}}$.

For $v \in Q$, $U(p_v,p_v) = T_{v,v}$, and $U(p_v)$ satisfies the equation

$$U(p_v)^2 - T_v \cdot U(p_v) + q_v T_{v,v} = 0 \tag{$*$}$$

by Lemma 3.30. By our assumption, it has two distinct roots in the residue field of $T^{\text{ß}}_{\mathscr{D}_y}$, and hence there are two distinct roots in $T^{\text{ß}}_{\mathscr{D}_y}$ by Hensel's lemma. By our choice of $\tilde{m}_{0,Q}$, $U(p_v)$ is the element α_v in $T^{\text{ß}}_{\mathscr{D}_y}$ which satisfies $(*)$, and lifts $\bar{\alpha}_v$. The claim is shown. □

8.2. **Twisted sheaves.** To control the central character when the infinity type (k,w) is not $((2,\ldots,2),0)$, a passage to the adjoint group $G^{\mathrm{ad}}_{\bar{\rho}}$ is necessary. We slightly modify the definitions of S_K and the sheaf $\bar{\mathscr{F}}_{(k,w)}$.

We return to the general setting. For a division algebra D over F with $q_D \leq 1$, let Z be the center of G_D. For an F-factorizable compact open subgroup U of $G(\mathbb{A}_{\mathbb{Q},f})$, we define

$$C_U = Z(\mathbb{Q})\backslash Z(\mathbb{A}_{\mathbb{Q}})/U \cap Z(\mathbb{A}_{\mathbb{Q},f}) \cdot Z(\mathbb{R}).$$

$Z(\mathbb{A}_{\mathbb{Q}})$ acts on $S_U = G_D(\mathbb{Q})\backslash G_D(\mathbb{A}_{\mathbb{Q}})/U \times K_\infty$ from the right. This action of $Z(\mathbb{A}_{\mathbb{Q}})$ induces a C_U-action on S_U.

Lemma 8.8. *We assume that there is a finite place y and a finite set S of finite places which satisfy conditions (1)-(3) of Proposition 4.9, and $U = U(y,S)$. Then the C_U-action on S_U is free.*

Proof of Lemma 8.8. Take an element $z \in Z(\mathbb{A}_{\mathbb{Q}_f})$ which represents a class c in C_U. We may assume that the finite part of z belongs to U by an approximation by an element of $F^\times$. By 4.10, there is an element $\epsilon \in F^\times$ such that $\epsilon^{-1} z \in U$. Then the class of z in C_U is trivial. □

Definition 8.9. *Notations are as in Lemma 8.8. For a prime ℓ, let $C_{U,\ell}$ be the ℓ-Sylow subgroup of C_U. We denote by $\tilde{S}_U$ the quotient of S_U by $C_{U,\ell}$.*

We view the pro-ℓ-part $\chi_{\mathrm{cycle},\ell}$ of χ_{cycle} as a continuous character of $F^\times\backslash(\mathbb{A}_{F,f})^\times$. We also view it as a continuous character of $G_D(\mathbb{Q})\backslash G_D(\mathbb{A}_{\mathbb{Q},f})$ via the reduced norm $G_D(\mathbb{A}_{\mathbb{Q},f}) \overset{\mathrm{N}_D}{\to} (\mathbb{A}_{F,f})^\times$. In particular $\chi_{\mathrm{cycle},\ell}$ can be viewed as a $G_D(\mathbb{A}_{\mathbb{Q},f})$-equivariant sheaf on $S = \varprojlim_U S_U$. Since $\chi_{\mathrm{cycle},\ell}$ takes values in $1 + \ell\mathbb{Z}_\ell$ and $\ell \geq 3$, the square root $\chi^{\frac{1}{2}}_{\mathrm{cycle},\ell}$ is defined as an ℓ-adic character.

Assume that U satisfies the condition of Lemma 8.8, and the sheaf $\bar{\mathscr{F}}^U_{(k,w),E_\lambda}$ is defined on S_U.

Definition 8.10. *For a prime $\ell \geq 3$,*

$$\bar{\mathscr{F}}^{U,\mathrm{tw}}_{(k,w)} = \bar{\mathscr{F}}^{U}_{(k,w)} \otimes \chi_{\mathrm{cycle},\ell}^{-\frac{w}{2}}.$$

By the definition of $\bar{\mathscr{F}}^{U,\mathrm{tw}}_{(k,w)}$, it is $G_D(\mathbb{A}^{\ell}_{\mathbb{Q},f})$-equivariant. We analyze the action of $Z(\mathbb{A}_{\mathbb{Q}_f})$ in more detail.

Lemma 8.11. *For a prime $\ell \geq 3$, let $\mathscr{Z}_\ell$ be the inverse image of $C_{U,\ell}$ by $Z(\mathbb{A}_{\mathbb{Q},f}) \to C_U$. Then the $\mathscr{Z}_\ell$-action on $\bar{\mathscr{F}}^{U,\mathrm{tw}}_{(k,w)}$ induces a structure of a C_U-equivariant sheaf on $\bar{\mathscr{F}}^{U,\mathrm{tw}}_{(k,w)}$, and $\bar{\mathscr{F}}^{U,\mathrm{tw}}_{(k,w)}$ descends to an o_{E_λ}-smooth sheaf $\tilde{\bar{\mathscr{F}}}^{U,\mathrm{tw}}_{(k,w)}$ on $\tilde{S}_U$.*

Proof of Lemma 8.11. We use the notations in §4.2. Let $\pi_\ell : \tilde{S}_\ell \to S_U$ be the $\overline{Z(\mathbb{Q})} \cap U_\ell \backslash U_\ell$-torsor, $V_{(k,w),E_\lambda}$ the representation of $G_D(\mathbb{Q}_\ell)$ defining $\bar{\mathscr{F}}^{U}_{(k,w),E_\lambda}$. We twist $V_{(k,w),E_\lambda}$ by $\chi_{\mathrm{cycle},\ell}^{-\frac{w}{2}}$, and define

$$V^{\mathrm{tw}}_{(k,w),E_\lambda} = V_{(k,w),E_\lambda} \otimes \chi_{\mathrm{cycle},\ell}^{-\frac{w}{2}}$$

as a continuous representation of $G_D(\mathbb{Q}_\ell)$. Since $\chi_{\mathrm{cycle},\ell}^{-\frac{w}{2}}$ is $o_{E_\lambda}^\times$-valued, $V_{(k,w),o_{E_\lambda}}$ gives an o_{E_λ}-lattice $V^{\mathrm{tw}}_{(k,w),o_{E_\lambda}}$ of $V^{\mathrm{tw}}_{(k,w),E_\lambda}$. $\mathscr{F}^{U,\mathrm{tw}}_{(k,w),E_\lambda}$ on S_U is obtained from π_ℓ and $G_D(\mathbb{Q}_\ell)$-representation $V^{\mathrm{tw}}_{(k,w),E_\lambda}$.

For $v|\ell$, we denote $\ker(o_{F_v}^\times \to k(v)^\times)$ by U_v^1. Since $\prod_{v|\ell} U_v^1$ acts trivially on $V^{\mathrm{tw}}_{(k,w),E_\lambda}$, any element $z \in Z(\mathbb{A}_{\mathbb{Q},f})$ such that the ℓ-component z_ℓ belongs to $\prod_{v|\ell} U_v^1$ acts trivially on $\mathscr{F}^{U,\mathrm{tw}}_{(k,w),E_\lambda}$. In particular an equivariant action of $Z(\mathbb{Q})\backslash Z(\mathbb{A}_{\mathbb{Q},f})/H$ is induced for

$$H = (\prod_{v|\ell} U_v \cap U_v^1) \cap Z(\mathbb{Q}_\ell) \cdot (U \cap Z(\mathbb{A}_{\mathbb{Q},f}))^{\ell}.$$

The ℓ-part of $Z(\mathbb{Q})\backslash Z(\mathbb{A}_{\mathbb{Q},f})/H$ is canonically isomorphic to $C_{U,\ell}$, and hence we obtain a $C_{U,\ell}$-action on $\bar{\mathscr{F}}^{U,\mathrm{tw}}_{(k,w),E_\lambda}$ which lifts the $C_{U,\ell}$-action S_U. By Lemma 8.8 and by descent, we have a sheaf $\tilde{\mathscr{F}}^{U,\mathrm{tw}}_{(k,w),E_\lambda}$ on $\tilde{S}_U$. Note that all classes of $C_{U,\ell}$ are represented by an element in $Z(\mathbb{A}^{\ell}_{\mathbb{Q},f})$, and hence the $\mathscr{Z}_\ell$-action on $\bar{\mathscr{F}}^{U,\mathrm{tw}}_{(k,w),E_\lambda}$ preserves the o_{E_λ}-structure $\bar{\mathscr{F}}^{U,\mathrm{tw}}_{(k,w)}$. Thus we have the o_{E_λ}-structure $\tilde{\bar{\mathscr{F}}}^{U,\mathrm{tw}}_{(k,w)}$. □

$$H^*(S_U,\ \bar{\mathscr{F}}^{U,\mathrm{tw}}_{(k,w)}) = H^*(S_U, \bar{\mathscr{F}}^{U}_{(k,w)}) \otimes \chi_{\mathrm{cycle},\ell}^{-\frac{w}{2}}$$

holds by the definition of $\bar{\mathscr{F}}^{U,\mathrm{tw}}_{(k,w)}$, and this equality is compatible with the action of the convolution algebra $H(G_D(\mathbb{A}^{\ell}_{\mathbb{Q},f}), U^\ell)_{o_{E_\lambda}}$, by regarding $\chi_{\mathrm{cycle},\ell}^{-\frac{w}{2}}$ as an ℓ-adic character of $G_D(\mathbb{A}^{\ell}_{\mathbb{Q},f})$.

We also obtain an action of $H(G_D(\mathbb{A}^{\ell}_{\mathbb{Q},f}), U^\ell)_{o_{E_\lambda}}$ on $H^*(\tilde{S}_U,\ \tilde{\bar{\mathscr{F}}}^{U,\mathrm{tw}}_{(k,w)})$ since the $C_{U,\ell}$-action on S_U comes from the center $Z(\mathbb{A}_{\mathbb{Q},f})$, and commutes with the Hecke correspondences.

8.3. **Twisted algebras and modules.** We go back to the situation in §8.1. We use the same notation as in §7.4. For our choice of the auxiliary place y and set S, $\Sigma = \Sigma_{\mathscr{D}_Q} \cup \{y\} \cup S$, H_{K^Σ} and $\tilde{m}_{\Sigma,\bar{\rho}}$ are the convolution algebra and the maximal ideal corresponding to $\bar{\rho}$, respectively. For $K'_Q = \ker \nu_{\mathscr{D}_y}|_{K_Q}$ (resp. $K'_{0,Q} = \ker \nu_{\mathscr{D}_y}|_{K_{0,Q}}$), we define $\tilde{M}_Q$ (resp. $\tilde{M}_{0,Q}$) by

$$\tilde{M}_Q = \mathrm{Hom}_{K_Q}(\nu_{\mathscr{D}_y}|_{K_Q}, H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_{K'_Q}, \bar{\mathscr{F}}^{K'_Q}_{(k,w)}))_{\tilde{m}_{\Sigma,\bar{\rho}}}$$

$$(\text{resp. } \tilde{M}_{0,Q} = \mathrm{Hom}_{K_{0,Q}}(\nu_{\mathscr{D}_y}|_{K_{0,Q}}, H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_{K'_{0,Q}}, \bar{\mathscr{F}}^{K'_{0,Q}}_{(k,w)}))_{\tilde{m}_{\Sigma,\bar{\rho}}}).$$

Similarly, $\tilde{M}^{\mathrm{tw}}_Q$ and $\tilde{M}^{\mathrm{tw}}_{0,Q}$ are defined as above by using $\bar{\mathscr{F}}^{\mathrm{tw}}_{(k,w)}$ instead of $\bar{\mathscr{F}}_{(k,w)}$.

The isomorphism $\tilde{M}^{\mathrm{tw}}_Q = \tilde{M}_Q \otimes \chi^{-\frac{w}{2}}_{\mathrm{cycle},\ell}$ (resp. $\tilde{M}^{\mathrm{tw}}_{0,Q} = \tilde{M}_{0,Q} \otimes \chi^{-\frac{w}{2}}_{\mathrm{cycle},\ell}$) respects the action of $H(G_{\bar{\rho}}(\mathbb{A}_{\mathbb{Q},f}), K^{\ell}_{0,Q})$ (resp. $H(G_{\bar{\rho}}(\mathbb{A}_{\mathbb{Q},f}), K^{\ell}_{Q})$) by viewing $\chi^{-\frac{w}{2}}_{\mathrm{cycle},\ell}$ as a one dimensional representation of the convolution algebra. This is also true for $\tilde{U}(p_v)$ and $\tilde{U}(p_v, p_v)$-operators. In particular for the $o_{\mathscr{D}}$-algebra $\tilde{T}^{\mathrm{tw}}_Q$ (resp. $\tilde{T}^{\mathrm{tw}}_{0,Q}$) generated by T^{tw}_v, $T^{\mathrm{tw}}_{v,v}$ ($v \notin \Sigma_{\mathscr{D}_Q} \cup \{y\} \cup S$), $U^{\mathrm{tw}}(p_v)$, $U^{\mathrm{tw}}(p_v, p_v)$ fo $v \in P^{\mathbf{u}}_{\mathscr{D}_Q} \setminus P^{\mathrm{exc}}$, $\tilde{U}^{\mathrm{tw}}(p_v)$, $\tilde{U}^{\mathrm{tw}}(p_v, p_v)$ for $v \in P^{\mathbf{n.o.}}$, $V^{\mathrm{tw}}(\delta)$ for $\delta \in \Delta_Q$ ($(-)^{\mathrm{tw}}$ denotes the Hecke operators defined by the twisted action on $\bar{\mathscr{F}}^{\mathrm{tw}}_{(k,w)}$), there is an $o_{\mathscr{D}}$-isomorphism

$$\tilde{T}_Q \simeq \tilde{T}^{\mathrm{tw}}_Q \quad (\text{resp. } \tilde{T}_{0,Q} \simeq \tilde{T}^{\mathrm{tw}}_{0,Q})$$

which maps the standard operators $(-)$ as above (T_v, for example) to $(-)^{\mathrm{tw}} \cdot \chi^{\frac{w}{2}}_{\mathrm{cycle},\ell}$. Let $\tilde{m}^{\mathrm{tw}}_Q$ (resp. $\tilde{m}^{\mathrm{tw}}_{0,Q}$) be the maximal ideal of $\tilde{T}^{\mathrm{tw}}_Q$ (resp. $\tilde{T}^{\mathrm{tw}}_{0,Q}$), T^{tw}_Q (resp. $T^{\mathrm{tw}}_{0,Q}$) the localization $(\tilde{T}^{\mathrm{tw}}_Q)_{\tilde{m}^{\mathrm{tw}}_Q}$ (resp. $(\tilde{T}^{\mathrm{tw}}_{0,Q})_{\tilde{m}^{\mathrm{tw}}_{0,Q}}$).

The corresponding T_Q (resp. $T_{0,Q}$)-module $M_Q = (\tilde{M}_Q)_{\tilde{m}_Q}$ (resp. $M_{0,Q} = (\tilde{M}_{0,Q})_{\tilde{m}_{0,Q}}$) and T^{tw}_Q (resp. $T^{\mathrm{tw}}_{0,Q}$)-module $M^{\mathrm{tw}}_Q = (\tilde{M}^{\mathrm{tw}}_Q)_{\tilde{m}^{\mathrm{tw}}_Q}$ (resp. $M^{\mathrm{tw}}_{0,Q} = (\tilde{M}^{\mathrm{tw}}_{0,Q})_{\tilde{m}^{\mathrm{tw}}_{0,Q}}$) are $o_{\mathscr{D}}$-free as in §7.4 by using Proposition 5.5, and the $o_{\mathscr{D}}$-algebra isomorphism

$$T_Q \simeq T^{\mathrm{tw}}_Q \quad (\text{resp. } T_{0,Q} \simeq T^{\mathrm{tw}}_{0,Q})$$

induces

$$M_Q \simeq M^{\mathrm{tw}}_Q \quad (\text{resp. } M_{0,Q} \simeq M^{\mathrm{tw}}_{0,Q}).$$

The same construction applies also to $T_{\mathscr{D}_y}$ and $M_{\mathscr{D}}$. $T^{\mathrm{tw}}_{\mathscr{D}_y}$ and $M^{\mathrm{tw}}_{\mathscr{D}}$ denotes the twisted algebra and the module which satisfy $T_{\mathscr{D}_y} \simeq T^{\mathrm{tw}}_{\mathscr{D}_y}$ and $M_{\mathscr{D}} \simeq M^{\mathrm{tw}}_{\mathscr{D}}$.

Proposition 8.12. *There is a natural isomorphism*

$$M_{\mathscr{D}} \simeq M_{0,Q},$$

which is compatible with the isomorphism $T_{\mathscr{D}} \simeq T_{\mathscr{D}_y} \simeq T_{0,Q}$ obtained by Proposition 7.12 and 8.7. The same is true for $M^{\mathrm{tw}}_{\mathscr{D}}$ and $M^{\mathrm{tw}}_{0,Q}$.

Proof of Proposition 8.12. We construct such an isomorphism by induction on $n = \sharp Q$. For $n = 0$ we take it as the identity. Assume that $Q = Q' \cup \{v\}$. The degeneracy maps $\mathrm{pr}_i : S_{K'_{0,Q}} \to S_{K'_{0,\emptyset}}$ for $i = 1, 2$ defined in §5.3 induce

$$H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_{K'_{0,Q'}}, \bar{\mathscr{F}}^{K'_{0,Q'}}_{(k,w)})^{\oplus 2} \overset{\mathrm{pr}_1 + \mathrm{pr}_2}{\longrightarrow} H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_{K'_{0,Q}}, \bar{\mathscr{F}}^{K'_{0,Q}}_{(k,w)}).$$

Thus we have a $\tilde{T}_Q$-homomorphism $\tilde{\beta}_Q : \tilde{M}^{\oplus 2}_{Q'} \to \tilde{M}_{0,Q}$.

By the decomposition of $H^{q_{\bar{\rho}}}$ in §4.4, we have an isomorphism

$$(I^0_{(k,w),\mathscr{D}_Q})^{\oplus 2^{q_{\bar{\rho}}}}_{\tilde{m}_{\Sigma,\bar{\rho}}} \simeq \tilde{M}_{0,Q} \otimes_{o_{\mathscr{D}}} \bar{E}_{\mathscr{D}},$$

which induce $\beta_{Q,E_{\mathscr{D}}} : \tilde{M}^{\oplus 2}_{Q',E_{\mathscr{D}}} \simeq \tilde{M}_{0,Q,E_{\mathscr{D}}}$ since the v-component of the representations which contibutes to $I^0_{(k,w),\mathscr{D}_Q}$ is spherical as in the proof of Proposition 8.7. In particular $\tilde{T}_Q = \tilde{T}_{Q'}[U(p_v)]$, and $\tilde{\beta}_Q$ is regarded as a $\tilde{T}_{Q'}[U(p_v)]$-homomorphism.

We need a special case of cohomological universal injectivity.

Sublemma 8.13. *Let D be a division algebra with $q_D \leq 1$ which is split at a finite place v. Assume that K is an F-factorizable compact open small subgroup of $G_D(\mathbb{A}_{\mathbb{Q}_f})$ with the v-component $K_v = \mathrm{GL}_2(o_{F_v})$. Then the kernel of the homomorphism defined by the degeneracy maps at v*

$$H^{q_D}(S_K,\ \bar{\mathscr{F}}^K_{(k,w),k_\lambda})^{\oplus 2} \overset{\mathrm{pr}_1^*+\mathrm{pr}_2^*}{\longrightarrow} H^{q_D}(S_{K\cap K_0(v)},\ \bar{\mathscr{F}}^{K\cap K_0(v)}_{(k,w),k_\lambda})$$

is annihilated by $T_v^2-(1+q_v)^2T_{v,v}$.

Proof of Sublemma 8.13. Consider the homomorphism

$$H^{q_D}(S_{K\cap K_0(v)},\ \bar{\mathscr{F}}_{(k,w),k_\lambda}) \xrightarrow{\gamma} H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w),k_\lambda})^{\oplus 2}$$

induced by the trace map. The composite

$$\delta : H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w),k_\lambda})^{\oplus 2} \overset{\mathrm{pr}_1^*+\mathrm{pr}_2^*}{\longrightarrow} H^{q_D}(S_{K\cap K_0(v)},\ \bar{\mathscr{F}}_{(k,w),k_\lambda}) \xrightarrow{\gamma} H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w),k_\lambda})^{\oplus 2}$$

written in the matrix form is

$$\delta = \begin{pmatrix} 1+q_v & T_v \\ T_{v,v}^{-1}\cdot T_v & 1+q_v \end{pmatrix},$$

which is shown in §10.2. If $\begin{pmatrix} a \\ b \end{pmatrix}$ is in the kernel of δ, $T_v^2-(1+q_v)^2T_{v,v}$ annihilates $\begin{pmatrix} a \\ b \end{pmatrix}$ by a direct calculation. The claim follows. □

By Sublemma 8.13, the image of $\tilde{\beta}_Q$ is an $o_{\mathscr{D}}$-direct summand. We already know that $\tilde{\beta}_{Q,E_{\mathscr{D}}}$ is an isomorphism. Thus $\tilde{\beta}_Q$ is an isomorphism.

The localization of $\tilde{M}_{Q'}$ at $\tilde{M}_{Q'}$ is isomorphic to $M_{\mathscr{D}}$ by our assumption on the induction. M_Q is isomorphic to the localization of $M_{Q'}^{\oplus 2}$ at the maximal ideal of $T_{Q'}[U(p_v)]$ generated by $m_{T_{Q'}}$ and $U(p_v)-\bar{\alpha}_v$. As in the proof of Proposition 8.7, $U(p_v)$ safisfies $f(U(p_v))=0$ for $f(X)=X^2-T_vX+q_vT_{v,v}$ and f has two distinct roots in $T_{Q'}$. For the root α_v of $f(X)$ which lifts $\bar{\alpha}_v$, M_Q is the submodule of $M_{Q'}^{\oplus 2}$ where $U(p_v)$ acts as α_v, which is identified with $M_{Q'}$. □

8.4. **Perfect complex argument.** In this subsection, we verify the axiom TW5 for M_Q.

If we set $K'_Q = \ker \nu_{\mathscr{D}_y}|_{K_Q(y,S)}$ and $K'_{0,Q} = \ker \nu_{\mathscr{D}_y}|_{K_{0,Q}(y,S)}$, then

$$\Delta_Q \overset{\alpha}{\simeq} \overline{Z(\mathbb{Q})}\cdot K'_Q\backslash K'_{0,Q}$$

holds. Here, $\alpha(\delta)$ for $\delta\in\Delta_Q$ is given by the class of $\begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$. In particular M_Q has a structure of an $o_{\mathscr{D}}[\Delta_Q]$-module. $\delta\in\Delta_Q$ acts by $V(\delta)$-operator.

Lemma 8.14. *$\Delta_Q\times C_{K'_Q,\ell}$-action on $S_{K'_Q}$ is free.*

Proof of Lemma 8.14. Let c be a class in $C_{K'_Q,\ell}$ represented by $z\in Z(\mathbb{A}_{\mathbb{Q},f})$. We may assume that the v-component of z belongs to $K_{11}(m_v)$ for $v\in Q$, and $z\in K'_Q$ by an approximation by an element of $F^\times$. For $\delta\in\Delta_Q$, $c(\delta')\in G_{\bar{\rho}}(\mathbb{A}_{\mathbb{Q},f})$ is the element with $c(\delta')_Q = \begin{pmatrix} \delta'^{-1} & 0 \\ 0 & \delta' \end{pmatrix}$ for some $\delta'\in\prod_{v\in Q} o_{F_v}^\times$ lifting δ.

Assume that $z\cdot c(\delta')$ has a fixed point in S_{K_Q}. By Lemma 4.10, and there is some unit $\epsilon\in o_F^\times$ such that $\epsilon^{-1}zc(\delta')\in K_Q$. This implies that $\delta=1$, and the class c of z is trivial. □

$C_{K'_{0,Q},\ell}$, $C_{K'_Q,\ell}$ are isomorphic to $C_{F,P^{\mathbf{u}}_{\mathscr{D}},\ell}$, the ℓ-part of the class group of F which is unramified outside $P^{\mathbf{u}}_{\mathscr{D}}$.

Proposition 8.15. *M_Q^{tw} is an $o_{\mathscr{D}}[C_{F,P_{\mathscr{D}}^{\mathrm{u}},\ell} \times \Delta_Q]$-free module. The same is true for M_Q.*

Proof of Proposition 8.15. For $F = \mathbb{Q}$ and $D = M_2(\mathbb{Q})$, this is essentially proved in [49], proposition 1. We may assume that D is a division algebra. We prove 8.15 by the perfect complex argument as in [15], §3.

For $\Sigma = \Sigma_{\mathscr{D}_Q} \cup \{y\} \cup S$, let $H = H(D^\times(\mathbb{A}_f^\Sigma),\ K^\Sigma)_{o_{\mathscr{D}}}$ be the convolution algebra, and $\tilde{m}_{\Sigma,\bar{\rho}}$ is the maximal maximal ideal corresponding to $\bar{\rho}$.

Since M_Q^{tw} is a direct summand of $H^{q_{\bar{\rho}}}(S_{K'_Q},\ \bar{\mathscr{F}}_{(k,w)}^{\mathrm{tw}})_{\tilde{m}_{\Sigma,\bar{\rho}}}$, it suffices to show the following claim. $\square$

Lemma 8.16. *$H^{q_{\bar{\rho}}}(S_{K'_Q},\ \bar{\mathscr{F}}_{(k,w)}^{\mathrm{tw}})_{\tilde{m}_{\Sigma,\bar{\rho}}}$ is $o_{\mathscr{D}}[C_{F,P_{\mathscr{D}}^{\mathrm{u}},\ell} \times \Delta_Q]$-free.*

Proof of Lemma 8.16. We need two sublemmas.

Sublemma 8.17. *Let A be a noetherian local algebra with the maximal ideal m_A and the residue field k_A, B a commutative A-algebra. Let $L^\cdot$ be a complex of B-modules bounded below. Assume the following conditions:*

1. *$L^\cdot$ defines an object of the derived category $D^b_{\mathrm{coh}}(A)$ of bounded A-complexes with finitely generated cohomology groups (in particular $L^\cdot \otimes_A^{\mathbb{L}} k_A$ is defined).*
2. *For a maximal ideal m of B above m_A, $H^i(L^\cdot \otimes_A^{\mathbb{L}} k_A) \otimes_B B_m$ is zero for $i \neq 0$. Here B_m is the localization of B at m.*

Then $H^i(L^\cdot) \otimes_B B_m$ is A-free for $i = 0$, and zero for $i \neq 0$.

Proof of Sublemma 8.17. This is lemma 3.2 of [15]. The proof is simple so we include it here. By (1), by replacing $L^\cdot$ by the canonical truncation $\tau_{\leq N} L^\cdot$ for a sufficiently large integer N, we may assume that $L^\cdot$ is bounded above. Since B_m is B-flat, we may assume that $B = B_m$. $H^i(L^\cdot \otimes_A^{\mathbb{L}} k_A)$ is zero except $i = 0$. By taking the minimal resolution of $L^\cdot$ as A-complexes, the claim follows. $\square$

Sublemma 8.18. *Let $\pi : X \to Y$ be an étale morphism between schemes of finite type over an algebraically closed field, which is a G-torsor for a finite group G, $\mathscr{F}$ a smooth Λ-sheaf on Y, where Λ is a finite $\mathbb{Z}_\ell$-algebra. Then $R\Gamma(X,\ \pi^*\mathscr{F})$ is a perfect complex of $\Lambda[G]$-modules, and*

$$R\Gamma(X,\ \pi^*\mathscr{F}) \otimes^{\mathbb{L}}_{\Lambda[G]} \Lambda[G]/I_G \to R\Gamma(Y,\ \mathscr{F})$$

holds. Here I_G is the augmentation ideal, and the map is induced by the trace map.

This is known (usually in the dual form) in any standard cohomology theory.

We go back to the proof of Lemma 8.16. $X = S_{K'_Q}$, $Y = \tilde{S}_{K'_{0,Q}}$, $\mathscr{F} = \tilde{\bar{\mathscr{F}}}_{(k,w)}$. We view X and Y as a scheme over $\mathbb{C}$. The natural projection $\pi : X \to Y$ is an étale G-torsor by Lemma 8.14 for $G = C_{F,P_{\mathscr{D}}^{\mathrm{u}},\ell} \times \Delta_Q$.

By the same argument as in Proposition 4.5, $R\Gamma(X,\ \pi^*\mathscr{F})$ is represented by a complex $L^\cdot$ of $H \otimes_{o_{\mathscr{D}}} o_{\mathscr{D}}[G]$-modules bounded below.

If $q_{\bar{\rho}} = 0$, Lemma 8.14 is clear since the G-action on X is free. We may assume that $q_{\bar{\rho}} = 1$. For $A = o_{\mathscr{D}}[G]$, and $B = A \otimes_{o_{\mathscr{D}}} H$, let m be a maximal ideal of B which is above m_A and $\tilde{m}_{\Sigma,\bar{\rho}}$. As in Lemma 5.4, the action of H on $H^q(Y,\ \mathscr{F} \otimes_{o_{\mathscr{D}}} k)$ is of residual type for $q = 0, 2$, and hence vanishes after the localization at m. By Sublemma 8.18, $H^q(L^\cdot \otimes_A A/m_A) \simeq H^q(Y,\ \mathscr{F} \otimes_{o_{\mathscr{D}}} k)$ for $q \in \mathbb{Z}$, and the localization $H^q(L^\cdot \otimes_A A/m_A)_m$ at m is zero except for $q = 1$. By Sublemma 8.17, $H^1(L^\cdot)_m = H^1(X,\ \pi^*\mathscr{F})_{\tilde{m}_{\Sigma,\bar{\rho}}}$ is $o_{\mathscr{D}}[G]$-free. $\square$

As a byproduct of the proof of Lemma 8.16, we have an isomorphism

$$(*)\qquad H^{q_{\bar\rho}}(S_{K'_Q},\ \bar{\mathscr{F}}^{\mathrm{tw}}_{(k,w)})_{\tilde m_{\Sigma,\bar\rho}} \otimes_{o_{\mathscr{D}}[C_{F,P^{\mathbf{u}}_{\mathscr{D}},\ell}\times\Delta_Q]} o_{\mathscr{D}}[C_{F,P^{\mathbf{u}}_{\mathscr{D}},\ell}\times\Delta_Q]/I_{C_{F,P^{\mathbf{u}}_{\mathscr{D}},\ell}\times\Delta_Q}$$

$$\xrightarrow{\ \sim\ } H^{q_{\bar\rho}}(\tilde S_{K'_{0,Q}},\ \bar{\mathscr{F}}^{\mathrm{tw}}_{(k,w)})_{\tilde m_{\Sigma,\bar\rho}}.$$

Here $I_{C_{F,P^{\mathbf{u}}_{\mathscr{D}},\ell}\times\Delta_Q}$ is the augmentation ideal of $o_{\mathscr{D}}[C_{F,P^{\mathbf{u}}_{\mathscr{D}},\ell}\times\Delta_Q]$. Similarly, by using the covering $S_{K'_Q}\to S_{K'_{0,Q}}$ instead of $S_{K'_Q}\to\tilde S_{K'_{0,Q}}$,

$$(**)\qquad H^{q_{\bar\rho}}(S_{K'_Q},\ \bar{\mathscr{F}}^{\mathrm{tw}}_{(k,w)})_{\tilde m_{\Sigma,\bar\rho}} \otimes_{o^{\mathrm{ass}}_{\mathscr{D}}[\Delta_Q]} o^{\mathrm{ass}}_{\mathscr{D}}[\Delta_Q]/I_{\Delta_Q} \xrightarrow{\ \sim\ } H^{q_{\bar\rho}}(S_{K'_{0,Q}},\ \bar{\mathscr{F}}^{\mathrm{tw}}_{(k,w)})_{\tilde m_{\Sigma,\bar\rho}}$$

holds for

$$o^{\mathrm{ass}}_{\mathscr{D}} = o_{\mathscr{D}}[C_{F,P^{\mathbf{u}}_{\mathscr{D}},\ell}],$$

and the augmentation ideal I_Q of $o^{\mathrm{ass}}_{\mathscr{D}}[\Delta_Q]$ over $o^{\mathrm{ass}}_{\mathscr{D}}$. It is easy to check that the isomorphisms $(*)$ and $(**)$ commute with the action of the standard Hecke operators. By using Proposition 8.12, we have

Proposition 8.19. *M^{tw}_Q is $o^{\mathrm{ass}}_{\mathscr{D}}[\Delta_Q]$-free, and $M^{\mathrm{tw}}_Q\otimes_{o^{\mathrm{ass}}_{\mathscr{D}}} o^{\mathrm{ass}}_{\mathscr{D}}[\Delta_Q]/I_Q \simeq M^{\mathrm{tw}}_{\mathscr{D}}$ as a T_Q-module. The same is true for M_Q and $M_{\mathscr{D}}$.*

8.5. **Construction of the system.** Finally we prove Theorem 8.1 by summarizing the result of the previous subsections.

Let $\chi^{\mathrm{Hecke}}_{\bar\rho}$ be the algebraic Hecke character of weight $2w$ attached to $\det\bar\rho$ (see Definition 6.10). $\chi=(\det\bar\rho)_{\mathrm{lift}}\cdot\chi^{-w}_{\mathrm{cycle},\ell}$ is the ℓ-adic representation attached to $\chi^{\mathrm{Hecke}}_{\bar\rho}$, where $(\det\bar\rho)_{\mathrm{lift}}$ is the Teichmüller lift of $\bar\rho$.

For a minimal deformation condition $\mathscr{D}$, $o^{\mathrm{ass}}_{\mathscr{D}}=o_{\mathscr{D}}[C_{F,\ell}]$ as in Theorem 8.1. $o^{\mathrm{ass}}_{\mathscr{D}}$ is seen as a universal deformation ring of $\det\bar\rho$ by the class field theory: for the character

$$\mu^{\mathrm{univ}}_{F,\ell}: G^{\mathrm{ab}}_F \xrightarrow{\ \sim\ } \pi_0(F^\times\backslash\mathbb{A}^\times_F)\twoheadrightarrow C_{F,\ell}\hookrightarrow (o_{\mathscr{D}}[C_{F,\ell}])^\times,$$

$\chi\cdot\mu^{\mathrm{univ}}_{F,\ell}$ is the universal deformation of $\det\bar\rho$, where the local conditions are finite at $v\nmid\ell$, and the restriction to I_{F_v} is $\chi|_{I_{F_v}}$ for $v|\ell$.

For the universal deformation ring $R_{\mathscr{D}}$ and the universal Galois representation $\rho^{\mathrm{univ}}_{\mathscr{D}}$, $\det\rho^{\mathrm{univ}}_{\mathscr{D}}$ satisfies the deformation conditions of $\det\bar\rho$ as above. Thus $R_{\mathscr{D}}$ has a structure of $o^{\mathrm{ass}}_{\mathscr{D}}$-algebra, and $\det\rho^{\mathrm{univ}}_{\mathscr{D}}$ is identified with $\chi\cdot\mu^{\mathrm{univ}}_F$. By the algebra homomorphism $\pi_{\mathscr{D}}:R_{\mathscr{D}}\to T_{\mathscr{D}}$ given by $\rho^{\mathrm{mod}}_{\mathscr{D}}$, $T_{\mathscr{D}}$ is also regarded as an $o^{\mathrm{ass}}_{\mathscr{D}}$-algebra.

For $Q\in\mathscr{X}_{\mathscr{D}}$, (T_Q,M_Q) is the pair of the Hecke ring and the Hecke module defined in §8.1 and §8.3, and ρ^{mod}_Q is the universal modular representation. By our construction, $\det\rho^{\mathrm{mod}}_Q$ is unramified at $v\in Q$. Thus we have an $o^{\mathrm{ass}}_{\mathscr{D}}$-algebra structure on T_Q as above. R_Q is the universal deformation ring of $\bar\rho$ of deformation type $\mathscr{D}_Q$ over $o^{\mathrm{ass}}_{\mathscr{D}}$ with the *determinant fixed* to $\chi\cdot\mu^{\mathrm{univ}}_F$. The universal deformation is denoted by ρ^{univ}_Q. The canonical algebra homomorphism $\pi_Q:R_Q\to T_Q$ is an $o^{\mathrm{ass}}_{\mathscr{D}}$-homomorphism, which is surjective since $\mathrm{trace}_{T_Q}(\mathrm{Fr}_v)$ for $v\notin\Sigma_{\mathscr{D}_Q}\cup\{y\}\cup S$ generate T_Q by Proposition 8.4.

Note that this $o^{\mathrm{ass}}_{\mathscr{D}}$-structure on T_Q is compatible with the $o^{\mathrm{ass}}_{\mathscr{D}}$-module structure on M_Q given by $M_Q\simeq M^{\mathrm{tw}}_Q$ and the $C_{F,\ell}$-action on M^{tw}_Q given in §8.2: the $C_{F,\ell}$-action is defined by the action of $Z(\mathbb{A}_{\mathbb{Q},f})$ on $S_{K'_Q}$, and $Z(\mathbb{A}_{\mathbb{Q},f})$-action yields the central character for any representation π which appear as a component of T_Q. The central character corresponds to the determinant (twisted by χ_{cycle}) by the Langlands correspondence, so the compatibility is shown.

For the natural projections $\alpha_Q : R_Q \twoheadrightarrow R_{\mathscr{D}}$, $\beta_Q : T_Q \twoheadrightarrow T_{\mathscr{D}}$, the diagram

$$\begin{array}{ccc} R_Q & \xrightarrow{\pi_Q} & T_Q \\ {\scriptstyle\alpha_Q}\downarrow & & {\scriptstyle\beta_Q}\downarrow \\ R_{\mathscr{D}} & \xrightarrow{\pi_{\mathscr{D}}} & T_{\mathscr{D}} \end{array}$$

is commutative: $\beta_Q \circ \pi_Q \circ \rho_Q^{\mathrm{univ}} = \beta_Q \circ \rho_Q^{\mathrm{mod}}$ and $\pi_{\mathscr{D}} \circ \alpha_Q \circ \rho_Q^{\mathrm{univ}} = \pi_{\mathscr{D}} \circ \rho_{\mathscr{D}}^{\mathrm{univ}} = \rho_{\mathscr{D}}^{\mathrm{mod}}$ have the same trace at Fr_v for $v \notin \Sigma_{\mathscr{D}_Q} \cup \{y\} \cup S$ (the standard Hecke operators at v). By the Chebotarev density theorem, the two representations have the same trace function on G_F, and hence are isomorphic by Lemma 7.11.

TW1 is clear from the definition of $X_{\mathscr{D}}$. We check axioms TW2-TW5 of Definition 2.1. By Proposition 3.8, $o_{\mathscr{D}}^{\mathrm{ass}}[\mathbb{Z}_\ell \times \Delta_v]$ is regarded as a versal hull of the unrestricted deformation of $\bar{\rho}|_{G_{F_v}}$ with a fixed determinant (we use the identification given there). We have a natural $o_{\mathscr{D}}^{\mathrm{ass}}$-algebra homomorphism

$$\tau_Q : o_{\mathscr{D}}^{\mathrm{ass}}[\Delta_Q] = \bigotimes_{v \in Q} o_{\mathscr{D}}^{\mathrm{ass}}[\Delta_v] \to R_Q,$$

which corresponds to the restriction $\prod_{v \in Q} \rho_Q^{\mathrm{univ}}|_{I_{F_v}}$. Thus R_Q has an $o_{\mathscr{D}}^{\mathrm{ass}}[\Delta_Q]$-algebra structure, and we view M_Q as an R_Q-module by f_Q. We use these structures in TW2.

By our normalization in Proposition 3.8 and Lemma 8.6,

$$\pi_Q \circ \tau_Q(\delta) = V(\delta) \quad (\delta \in \Delta_Q).$$

So $o_{\mathscr{D}}^{\mathrm{ass}}[\Delta_Q] \overset{\pi_Q \circ \tau_Q}{\to} T_Q$ is equal to the homomorphism $o_{\mathscr{D}}^{\mathrm{ass}}[\Delta_Q] \to T_Q$ obtained by the $V(\delta)$-operators. It is clear that $R_Q/I_Q R_Q \simeq R$, since $R_Q/I_Q R_Q$ is the maximal quotient of R_Q where ρ_Q^{univ} is unramified at Q by Faltings' theorem 3.8. TW3 is verified.

We regard M_Q as an R_Q-module by π_Q, and the action of $o_{\mathscr{D}}^{\mathrm{ass}}[\Delta_Q]$ through R_Q is the same as the $C_{F,\ell} \times \Delta_Q$-action geometrically constructed on M_Q^{tw} by the isomorphism $M_Q \simeq M_Q^{\mathrm{tw}}$. By Proposition 8.19, $M_Q/I_Q M_Q \simeq M_{\mathscr{D}}$, and TW4 is satisfied.

Finally we check TW5. By Proposition 8.15, M_Q is a free $o_{\mathscr{D}}^{\mathrm{ass}}[\Delta_Q]$-module. The $o_{\mathscr{D}}^{\mathrm{ass}}[\Delta_Q]$-rank of M_Q is the $o_{\mathscr{D}}^{\mathrm{ass}}$-rank of $M_Q/I_Q M_Q$, and which is non-zero and independent of Q by Proposition 8.19. Thus TW5 is satisfied for $(R_Q, M_Q)_{Q \in \mathscr{X}_{\mathscr{D}}}$.

Remark 8.20. *Similarly, one has a Taylor-Wiles system $(R_{Q,\chi},\ \tilde{M}_Q)_{Q \in \mathscr{X}_{\mathscr{D}}}$ for the deformation ring $R_{\mathscr{D},\chi}$ with determinant fixed to χ. Here $\tilde{M}_Q$ is the image of M_Q in $H^{q_{\bar{\rho}}}(\tilde{S}_{K'_Q},\ \bar{\mathscr{F}}_{(k,w)}^{\mathrm{tw}})_{\tilde{m}_{\Sigma,\bar{\rho}}} \otimes \chi_{\mathrm{cycle},\ell}^{\frac{w}{2}}$ in the notation of §8.4.*

9. The minimal case

The main purpose of this section is to prove the following theorem.

Theorem 9.1. *Let F be a totally real number field, $\bar{\rho} : G_F \to \mathrm{GL}_2(k_\lambda)$ an absolutely irreducible mod ℓ-representation. We take a minimal deformation condition $\mathscr{D}$, and assume the following conditions.*

1. *$\ell \geq 3$, and $\bar{\rho}|_{F(\zeta_\ell)}$ is absolutely irreducible. When $\ell = 5$, the following case is excluded: the projective image $\bar{G}$ of $\bar{\rho}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, and the mod ℓ-cyclotomic character $\bar{\chi}_{\mathrm{cycle}}$ factors through $G_F \to \bar{G}^{\mathrm{ab}} \simeq \mathbb{Z}/2$ (in particular $[F(\zeta_5) : F] = 2$).*
2. *For $v|\ell$, the deformation condition for $\bar{\rho}|_{G_{F_v}}$ is either nearly ordinary or flat (cf. 3.5). When the condition is nearly ordinary (resp. flat) at v, we assume that $\bar{\rho}|_{G_{F_v}}$ is G_{F_v}-distinguished (resp. F_v is absolutely unramified).*

(3) *There is a minimal modular lifting π of $\bar\rho$ over $o_{\mathscr{D}}$ as in Definition 6.11.*
(4) *Hypothesis 6.7 is satisfied.*

Then the universal deformation ring $R_{\mathscr{D}}$ of $\bar\rho$ with the deformation condition $\mathscr{D}$ is a complete intersection of relative dimension zero over $o_{\mathscr{D}}^{\mathrm{ass}} = o_{\mathscr{D}}[C_{F,\ell}]$, and $M_{\mathscr{D}}$ is a free $R_{\mathscr{D}}$-module. In particular $R_{\mathscr{D}}$ is isomorphic to the Hecke algebra $T_{\mathscr{D}}$ attached to $\bar\rho$. Here $C_{F,\ell}$ is the ℓ-part of the class group of F.

Note that there is an exceptional case which does not happen in [51].

9.1. **Preliminary on finite subgroups of** $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$**.** Let ℓ be a prime. Consider an absolutely irreducible representation $\rho : G \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ of a finite group G. By the classification of finite subgroups of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, the projective image $\bar G$ of ρ is in the following list:

(1) $\ell \geq 5$, $\bar G \simeq A_4$. $\bar G^{\mathrm{ab}}$ is isomorphic to $\mathbb{Z}/3$.
(2) $\ell \geq 5$, $\bar G \simeq S_4$. $\bar G^{\mathrm{ab}}$ is isomorphic to $\mathbb{Z}/2$.
(3) $\ell \geq 7$, $\bar G \simeq A_5$. $\bar G^{\mathrm{ab}}$ is trivial.
(4) (Dihedral case) $\ell \geq 2$, $\bar G \simeq D_{2n}$, where n is prime to ℓ, $n \geq 2$, and D_{2n} is a dihedral group of order $2n$. $\bar G^{\mathrm{ab}}$ is isomorphic to $\mathbb{Z}/2$ (resp. $\mathbb{Z}/2 \times \mathbb{Z}/2$) if n is odd (resp. even).
(5) (PSL_2-case) $\ell^n \geq 3$, $\bar G \simeq \mathrm{PSL}_2(\mathbb{F}_{\ell^n})$. $\bar G^{\mathrm{ab}}$ is trivial except for $\mathbb{F}_{\ell^n} = \mathbb{F}_3$, and is isomorphic to $\mathbb{Z}/3$ in the exceptional case.
(6) (PGL_2-case) $\ell^n \geq 3$, $\bar G \simeq \mathrm{PGL}_2(\mathbb{F}_{\ell^n})$. $\bar G^{\mathrm{ab}} \simeq \mathbb{Z}/2$.

The above 6 cases are exclusive to one another. In the case of (5) (resp. (6)), $\bar G$ is $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$-conjugate to the standard embedding of $\mathrm{PSL}_2(\mathbb{F}_{\ell^n})$ (resp. $\mathrm{PGL}_2(\mathbb{F}_{\ell^n})$) via the embedding $\mathbb{F}_{\ell^n} \hookrightarrow \overline{\mathbb{F}}_\ell$.

When $\ell \geq 3$, $\bar G$ is classified into (5) or (6) if and only if ℓ divides the order of $\bar G$.

Assume that $\ell \geq 3$, and let $\rho : G \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be an absolutely irreducible representation. Then the following conditions are equivalent:

- ρ is monomial.
- $\mathrm{ad}^0\rho$ is absolutely reducible.
- The projective image $\bar G$ of ρ is in the dihedral case.

If one of the above conditions holds, then for an index 2-subgroup H of G such that ρ is induced from a character χ of H, $\mathrm{ad}^0\rho \simeq \nu \oplus \mathrm{Ind}_H^G \chi/\chi^c$. Here ν is the character defined as the composition of $G \to G/H \simeq \{\pm 1\}$, and χ^c is the c-twist of χ for the generator c of G/H. $\mathrm{Ind}_H^G \chi/\chi^c$ is absolutely reducible if and only if $\bar G \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

We prepare several group theoretical lemmas which are used in the Chebotarev density argument in §9.2 .

Lemma 9.2. *Assume that $\ell \geq 3$. Let $\rho : G \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be an absolutely irreducible representation of a finite group G, and H a normal subgroup of G such that G/H is cyclic.*

Then there is an element $\sigma \in H$ of order prime to ℓ such that $\rho(\sigma)$ is regular and semi-simple.

Proof of Lemma 9.2. Assume that $\rho(h)$ has the same eigenvalues for any $h \in H$. Consider the H-representation $\mathrm{ad}^0\rho|_H$. Then the H-action factors through an ℓ-group, since the all eigenvalues of any element of H are one. So $\mathrm{ad}^0\rho|_H$ has only trivial constituents.

G/H acts on the invariant subspace $(\mathrm{ad}^0\rho)^H \neq \{0\}$. Since G/H is abelian, the G-action on $(\mathrm{ad}^0\rho)^H$ is absolutely reducible, and hence $\mathrm{ad}^0\rho$ is also absolutely reducible. Since $\ell \neq 2$, ρ is monomial by the absolute reducibility of $\mathrm{ad}^0\rho$. Since the projective image $\bar G$ of G has order prime to ℓ, the H-action on $\mathrm{ad}^0\rho$ is semi-simple. H acts trivially on $\mathrm{ad}^0\rho$, and hence $\bar G$ is a quotient of G/H. This is impossible since $\bar G$ is dihedral, and G/H is cyclic.

So there is an element $h \in H$ such that $\rho(h)$ is regular and semi-simple. By replacing h by a suitable ℓ^m-th power σ for some $m \geq 0$, the order of σ is prime to ℓ, and σ is regular and semi-simple. $\square$

Remark 9.3. *By the proof of 9.2, the consequence remains true under the assumption that G/H is abelian unless $\bar{G}$ is dihedral and abelian.*

In the dihedral case, we need a slightly stronger result than Lemma 9.2.

Lemma 9.4. *Assume that $\ell \geq 3$. Let $\rho : G \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be a monomial representation of a finite group G, H a subgroup of G such that $\rho|_H$ is absolutely irreducible. Then for any non-zero absolutely irreducible subspace N of $\mathrm{ad}^0 \rho$, there is an element $\sigma \in H$ of order prime to ℓ such that $\rho(\sigma)$ is regular and semi-simple, and $N^{<\sigma>} \neq \{0\}$.*

Proof of Lemma 9.4. This is essentially [51], lemma 1.12.

By the assumption, the restriction of ρ to H is absolutely irreducible. Since the H-action on $\mathrm{ad}^0 \rho$ is absolutely reducible, it is also monomial. We may assume that $H = G$, since it suffices to find $\sigma \in H$ for a non-zero absolutely irreducible H-subrepresentation N' of N.

First assume that the projective image $\bar{G}$ of ρ is not abelian. Then $\mathrm{ad}^0 \rho$ is isomorphic to $\nu \oplus V$, where ν is a character of order 2 such that ρ is induced from a character χ of $\ker \nu$, and V is irreducible. When N is the subspace which is isomorphic to V, we take any element $\sigma \in G$ of order prime to ℓ such that $\nu(\sigma) \neq 1$. When N is the subspace where G acts by ν, one takes any element $\sigma \in \ker \nu$ of order prime to ℓ such that $\chi/\chi^c(\sigma) \neq 1$.

Finally we treat the case when $\bar{G}$ is dihedral and abelian, that is, $\bar{G} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$. $\mathrm{ad}^0 \rho \simeq \alpha \oplus \beta \oplus \alpha \cdot \beta$ is isomorphic to the direct sum of the non-trivial characters of $\bar{G}$. By replacing the role of characters if necessary, we may assume that N is the subspace where G acts by α. Any element σ of order prime to ℓ such that $\alpha(\sigma) = 1$ and $\beta(\sigma) \neq 1$ satisfies the desired condition. $\square$

Lemma 9.5. *Assume that $\ell \geq 3$. Let $\rho : G \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be a representation of a finite group G, H a normal subgroup of G of ℓ-power index.*

1. *ρ is absolutely irreducible if and only if $\bar{\rho}|_H$ is absolutely irreducible.*
2. *Assume that ρ is absolutely irreducible. Then the projective image $\bar{H}$ of $\rho|_H$ is equal to the projective image $\bar{G}$ of ρ except when $\ell = 3$, $\bar{G} \simeq \mathrm{PSL}_2(\mathbb{F}_3)$, and $\bar{H} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.*

Proof of Lemma 9.5. For (1), it suffices to show that ρ is absolutely reducible if $\rho|_H$ is absolutely reducible. Let X be the set of all constituants of $\rho|_H$. The cardinality of X is at most 2. $g \in G$ acts on X by sending χ to the g-twist χ^g. Since G/H is an ℓ-group and $\ell \geq 3$, the action is trivial, and any element χ in X is the restriction of a character $\tilde{\chi}$ of G. Assume that χ appears as a subrepresentation of $\rho|_H$. Then $(\rho \otimes \tilde{\chi}^{-1})^H \neq \{0\}$. Since G/H is an ℓ-group, $(\rho \otimes \tilde{\chi}^{-1})^G$ is non-zero, and hence ρ is absolutely reducible.

For (2), note that $\bar{H} \neq \bar{G}$ implies that the order of $\bar{G}^{\mathrm{ab}}$ is divisible by ℓ. Then the result follows from the classification of subgroups of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$. $\square$

Proposition 9.6. *Let $\rho : G \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be an absolutely irreducible representation of a finite group, $\ell \neq 2$, $\mu : G \to \overline{\mathbb{F}}_\ell^\times$ a character of even order. Then*

$$H^1(G/(\ker(\mathrm{ad}^0 \rho \otimes \mu) \cap \ker \mu),\ \mathrm{ad}^0 \rho \otimes \mu) = 0$$

except when

- *$\ell = 5$, the projective image of ρ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, and μ factors through the character of order 2 of $\mathrm{PGL}_2(\mathbb{F}_5)$.*

Proof of Proposition 9.6. Note that $(\ker(\mathrm{ad}^0\rho\otimes\mu)\cap\ker\mu = (\ker\mathrm{ad}^0\rho)\cap\ker\mu$. Define subgroups H' and H of G by $\ker\mathrm{ad}^0\rho$ and $\ker\mathrm{ad}^0\rho\cap\ker\mu$ respectively. By the Hochshild-Serre spectral sequence,

$$0\longrightarrow H^1(G/H',(\mathrm{ad}^0\rho\otimes\mu)^{H'/H})\longrightarrow H^1(G/H,\mathrm{ad}^0\rho\otimes\mu)\longrightarrow H^1(H'/H,\mathrm{ad}^0\rho\otimes\mu)$$

is exact. If $\mu|_{H'}$ is non-trivial, then $H^1(G/H,\mathrm{ad}^0\rho\otimes\mu)$ is zero since the order of $H'/H\simeq\mu(H')$ is prime to ℓ, and $(\mathrm{ad}^0\rho\otimes\mu)^{H'/H}$ vanishes.

So we may assume that μ is a character of $\bar{G}=G/H'$, that is, a character of the projective image of ρ.

If the order of $\bar{G}$ is prime to ℓ, then the statement of 9.6 is clear, and hence we may assume that $\bar{G}$ contains an element of order ℓ. By the classification, $\bar{G}$ is either $\mathrm{PSL}_2(\mathbb{F}_{\ell^n})$ or $\mathrm{PGL}_2(\mathbb{F}_{\ell^n})$ for some $n\geq 1$. Since the order of μ is even, it follows that $\bar{G}$ is conjugate to the standard embedding of $\mathrm{PGL}_2(\mathbb{F}_{\ell^n})$, the order of μ is 2, μ is obtained from $(\det)^{\frac{\ell^n-1}{2}}$ of $\mathrm{GL}_2(\mathbb{F}_{\ell^n})$, and $\mathrm{ad}^0\rho\otimes\mu$ is isomorphic to $\mathrm{Sym}^2\otimes(\det)^{\frac{\ell^n-1}{2}-1}$ as $\mathrm{PGL}_2(\mathbb{F}_\ell)$-representations over $\overline{\mathbb{F}}_\ell$.

Then we apply the following lemma.

Lemma 9.7. *Assume that ℓ is an odd prime. Then*

$$H^1(\mathrm{SL}_2(\mathbb{F}_{\ell^n}),\mathrm{Sym}^2)=0$$

except for $\mathbb{F}_{\ell^n}=\mathbb{F}_5$.

This follows from [7], p. 185, Table (4.5) if $\mathbb{F}_{\ell^n}\neq\mathbb{F}_3$. In the $\mathbb{F}_3$-case, it is shown in [51], p. 478.

The vanishing of the cohomology now follows from two exact sequences

$$0\longrightarrow H^1(\mathbb{F}_{\ell^n}^\times,(\mathrm{Sym}^2\otimes(\det)^{\frac{\ell^n-1}{2}-1})^{\mathrm{SL}_2(\mathbb{F}_{\ell^n})})\longrightarrow H^1(\mathrm{GL}_2(\mathbb{F}_{\ell^n}),\mathrm{Sym}^2\otimes(\det)^{\frac{\ell^n-1}{2}-1})$$

$$\longrightarrow H^1(\mathrm{SL}_2(\mathbb{F}_{\ell^n}),\mathrm{Sym}^2\otimes(\det)^{\frac{\ell^n-1}{2}-1})^{\mathbb{F}_\ell^\times}$$

$$0\longrightarrow H^1(\mathrm{PGL}_2(\mathbb{F}_{\ell^n}),(\mathrm{Sym}^2\otimes(\det)^{\frac{\ell^n-1}{2}-1})^{\mathbb{F}_{\ell^n}^\times})\longrightarrow H^1(\mathrm{GL}_2(\mathbb{F}_{\ell^n}),\mathrm{Sym}^2\otimes(\det)^{\frac{\ell^n-1}{2}-1})$$

and Lemma 9.7, except for the $\mathbb{F}_5$-case.

□

As for the application to elliptic curves, the exceptional case in Proposition 9.6 does not happen.

Proposition 9.8. *Assume that G contains an element c of order 2 such that $\mu(c)=-1$, and $\rho(G)$ is a subgroup of $\mathrm{GL}_2(\mathbb{F}_5)$. Then the exceptional case in Proposition 9.6 does not occur for the triplet (G,ρ,μ).*

Proof. μ is identified with $(\det)^2$. The equation $\mu(c)=-1=(\det\rho(c))^2$ is impossible since $\det\rho(c)\in\{\pm 1\}$. □

Remark 9.9. *It can be shown that $H^1(\mathrm{GL}_2(\mathbb{F}_{\ell^n}),\mathrm{Sym}^2\otimes(\det)^{-1})$ is zero if $\ell\neq 2$. By [7], $H^1(\mathrm{SL}_2(\mathbb{F}_5),\mathrm{Sym}^2)$ is one dimensional. Using this, one shows that $H^1(\mathrm{PGL}_2(\mathbb{F}_5),\mathrm{Sym}^2\otimes\det)$ is one dimensional.*

9.2. **Chebotarev density argument.** We formulate an application of the Chebotarev density theorem following [51], proposition 1.11.

Definition 9.10. *Let F be a global field, M a finite $\mathbb{Z}_\ell[G_F]$-module given by $\rho_M : G_F \to \mathrm{Aut}_{\mathbb{Z}_\ell} M$, and H an open normal subgroup of G_F. For a finite Galois extension F' of F, let $F'_{(M,H)}$ be the Galois extension which corresponds to $(\ker \rho_M|_{F'}) \cap H$. A cohomology class x in $H^1(F,\ M)$ is exceptional for (M, H) over F' if the restriction $x_{F'}$ of x to F' belongs to $H^1(\mathrm{Gal}(F'_{(M,H)}/F'),\ M) = \ker(H^1(F',\ M) \to H^1(F'_{(M,H)},\ M))$.*

Proposition 9.11. *Notations are as in Definition 9.10. Take a non-zero cohomology class x in $H^1(F,\ M)$. Assume the following conditions.*

1. *F' is a Galois extension of F, and x is not exceptional for (M, H) over F' (cf. 9.10).*
2. *For any irreducible $\mathbb{Z}_\ell[G_F]$-submodule N of M, a non-trivial element $g_N \in \mathrm{Gal}(F'_{(M,H)}/F')$ is given. The order of g_N is prime to ℓ, g_N belongs to the image of $G_{F'} \cap H$, and $N^{<g_N>} \neq \{0\}$.*

Then there is a finite place v of F that satisfies:

- *v is of degree 1 and splits completely in F'.*
- *M is unramified at v, and the restriction of x to $H^1(F_v,\ M)$ is not zero.*
- *Fr_v acts on M by a G_F-conjugate of g_N for some non-zero irreducible submodule N. In particular the image of Fr_v in G_F belongs to $G_{F'} \cap H$.*

Such a place v exists with a positive density.

Proof of Proposition 9.11. $L = F'_{(M,H)}$, $\mathscr{G} = \mathrm{Gal}(L/F')$. The image y of $x_{F'}$ under the restriction map $H^1(F',\ M) \to H^1(L,\ M)$ does not vanish by assumption (1) of 9.11. We regard y as a non-zero G_F-equivariant homomorphism $c(y) : G_L \to M$. Let L_y be the abelian extension corresponding to the kernel of $c(y)$ over L. $\mathrm{Gal}(L_y/L)$ with the conjugate action of G_F is identified with $c(y)(G_L)$ as $\mathbb{Z}_\ell[G_F]$-modules.

For a non-zero irreducible $\mathbb{Z}_\ell[G_F]$-submodule N in $c(y)(G_L)$, take the element g_N in $\mathscr{G}$ of order α that is given for N in 9.11 (2), and an element $\tilde{\sigma}$ in $\mathrm{Gal}(L_y/F')$ which lifts g_N and has the same order as g_N (this is possible since α is prime to $[L_y : L]$). $g_N \in H$, and $N^{<g_N>} \neq \{0\}$. Choose a non-zero element $\tilde{\tau}$ in $\mathrm{Gal}(L_y/L)$ that belongs to $N^{<g_N>}$ by the identification $\mathrm{Gal}(L_y/L) \simeq c(y)(G_L)$.

Since $\tilde{\tau}$ belongs to $N^{<g_N>}$, $\tilde{\tau}$ is fixed by $\tilde{\sigma}$. This implies that $\tilde{\tau}$ commutes with $\tilde{\sigma}$. We define a conjugacy class C in $\mathrm{Gal}(L_y/F)$ as the class of $\tilde{\sigma}\tilde{\tau}$. C is contained in $\mathrm{Gal}(L_y/F')$ since F' is Galois over F.

Take a finite place v of F of degree one such that M is unramified at v, and $\mathrm{Fr}_v \in C$ by the Chebotarev density theorem. v splits completely in F' since the image of Fr_v in $\mathrm{Gal}(F'/F)$ is trivial. $c(y)(\mathrm{Fr}_v^\alpha)$ is in the $\mathrm{Gal}(L_y/F')$-orbit of $\tilde{\tau}^\alpha$, which is non-zero by the construction, and hence the restriction of x to $H^1(F_v, M)$ is non-zero. □

Corollary 9.12. *Notations are as in Definition 9.10. Assume moreover that M is a $k_\lambda[G_F]$-module for some finite field k_λ, and the following conditions.*

1. *F' is a Galois extension of F, and x is not exceptional for (M, H) over F' (cf. 9.10).*
2. *For any irreducible $k_\lambda[G_F]$-submodule N of M, a non-trivial element $g_N \in \mathrm{Gal}(F'_{(M,H)}/F')$ is given. The order of g_N is prime to ℓ, g_N belongs to the image of $G_{F'} \cap H$, and $N^{<g_N>} \neq \{0\}$.*

Then there is a finite place v of F that satisfies:

- *v is of degree 1 and splits completely in F'.*

- *M is unramified at v, and the restriction of x to $H^1(F_v,\ M)$ is not zero.*
- *Fr_v acts on M by a G_F-conjugate of g_N for some non-zero irreducible submodule N. In particular the image of Fr_v in G_F belongs to $G_{F'} \cap H$.*

Proof of corollary 9.12. We construct g_N as in Proposition 9.11 for any non-zero irreducible $\mathbb{F}_\ell[G_F]$-submodule N of M. Let W be the image of $N \otimes_{\mathbb{F}_\ell} k_\lambda$ in M, which is non-zero since it contains N. We take a non-zero $k_\lambda[G_F]$-irreducible submodule N' in W, and the element $g_{N'}$ given for this N' in 9.12 (2). Since the order of $g_{N'}$ is prime to ℓ, $(N \otimes_{\mathbb{F}_\ell} k_\lambda)^{<g_{N'}>} \to W^{<g_{N'}>}$ is surjective. $N'^{<g_{N'}>} \subset W^{<g_{N'}>}$, and hence $(N \otimes_{\mathbb{F}_\ell} k_\lambda)^{<g_{N'}>} = N^{<g_{N'}>} \otimes_{\mathbb{F}_\ell} k_\lambda$ is non-zero. By taking $g_{N'}$ as g_N, Proposition 9.11, (2) is satisfied, and Proposition 9.11 is applied. $\square$

9.3. **Proof of Theorem 9.1.** We prove Theorem 9.1. We use the same notation as in §8.5. We have already constructed a Taylor-Wiles system $\{R_Q,\ M_Q\}_{Q \in \mathscr{X}_{\mathscr{D}}}$ for $(R_{\mathscr{D}}, M_{\mathscr{D}})$ over $o_{\mathscr{D}}^{\mathrm{ass}}$.

Recall that

$$\mathrm{Hom}_{k_\lambda}(m_{R_Q}/(m_{R_Q}^2, m_{o_{\mathscr{D}}^{\mathrm{ass}}}),\ k_\lambda) \ = \ H^1_{\mathscr{D}_Q}(F,\ \mathrm{ad}^0 \bar{\rho}),$$

for the universal deformation ring $R_{\mathscr{D}_Q}$, and $R_{\mathscr{D}_Q}$ is generated by $\dim_{k_\lambda} H^1_{\mathscr{D}_Q}(F,\ \mathrm{ad}^0 \bar{\rho})$ many elements over o_{E_λ}. Here

$$H^1_{\mathscr{D}_Q}(F,\ \mathrm{ad}^0 \bar{\rho}) = \ker(H^1(F,\ \mathrm{ad}^0 \bar{\rho}) \longrightarrow \bigoplus_{v \in |F|_f} H^1(F_v,\ \mathrm{ad}^0 \bar{\rho})/L'_v)$$

by Proposition 3.35, with the subspace $L'_v = H^1_{\mathrm{def}_{\mathscr{D}_Q}(v)}(F_v,\ \mathrm{ad}^0 \bar{\rho})$ for any finite place v.

For the Tate dual $\mathrm{ad}^0 \bar{\rho}^* = \mathrm{ad}^0 \bar{\rho}(1)$ of $\mathrm{ad}^0 \bar{\rho}$, the dual Selmer groups are defined by

$$H^1_{\mathscr{D}^*}(F,\ \mathrm{ad}^0 \bar{\rho}(1)) = \ker(H^1(F,\ \mathrm{ad}^0 \bar{\rho}(1)) \longrightarrow \bigoplus_{v \in |F|_f} H^1(F_v,\ \mathrm{ad}^0 \bar{\rho}(1))/L^*_v)$$

$$H^1_{\mathscr{D}_Q{}^*}(F,\ \mathrm{ad}^0 \bar{\rho}(1)) = \ker(H^1(F,\ \mathrm{ad}^0 \bar{\rho}(1)) \longrightarrow \bigoplus_{v \in |F|_f} H^1(F_v,\ \mathrm{ad}^0 \bar{\rho}(1))/L'^*_v)$$

Here L^*_v (resp. L'^*_v) is the annihilator of $H^1_{\mathrm{def}_{\mathscr{D}'}(v)}(F_v,\ \mathrm{ad}^0 \bar{\rho})$ (resp. $H^1_{\mathrm{def}_{\mathscr{D}_Q}(v)}(F_v,\ \mathrm{ad}^0 \bar{\rho})$) by the local duality.

To prove Theorem 9.1, it is sufficient to find a non-empty subset $\mathscr{X} \subset \mathscr{X}_{\mathscr{D}}$ which satisfies the assumptions of the complete intersection-freeness criterion (Theorem 2.3) under the minimality of $\mathscr{D}$. For this, it suffices to show the following proposition.

Proposition 9.13. *Assume that $\mathscr{D}$ is minimal. For any integer $n \geq 1$, there is an element $Q \in \mathscr{X}_{\mathscr{D}}$ such that*

1. $q_v \equiv 1 \mod \ell^n$ *for* $v \in Q$,
2. $\dim_{k_\lambda} H^1_{\mathscr{D}_Q}(F,\ \mathrm{ad}^0 \bar{\rho}) \leq \sharp Q$,
3. $\sharp Q = \dim_{k_\lambda} H^1_{\mathscr{D}^*}(F,\ \mathrm{ad}^0 \bar{\rho}(1))$

hold.

Lemma 9.14. *Assume that $\mathscr{D}$ is minimal.*

1. *The inequality*

$$\dim_{k_\lambda} H^1_{\mathscr{D}}(F,\ \mathrm{ad}^0 \bar{\rho}) \leq \dim_{k_\lambda} H^1_{\mathscr{D}^*}(F,\ \mathrm{ad}^0 \bar{\rho}(1))$$

   holds.
2. *For any element Q of $\mathscr{X}_{\mathscr{D}}$,*

$$\dim_{k_\lambda} H^1_{\mathscr{D}_Q}(F,\ \mathrm{ad}^0 \bar{\rho}) \leq \dim_{k_\lambda} H^1_{\mathscr{D}_Q^*}(F,\ \mathrm{ad}^0 \bar{\rho}(1)) + \sharp Q.$$

Proof of Lemma 9.14. We prove (1). By the formula in [51], proposition 1.6,

$$\dim_{k_\lambda} H^1_{\mathscr{D}}(F,\ \mathrm{ad}^0\bar{\rho}) - \dim_{k_\lambda} H^1_{\mathscr{D}^*}(F,\ \mathrm{ad}^0\bar{\rho}(1))$$
$$= \sum_{v\in\Sigma_{\mathscr{D}}} h_v + \sum_{v\in I_F} h_v + \dim_{k_\lambda} H^0(F,\ \mathrm{ad}^0\bar{\rho}) - \dim_{k_\lambda} H^0(F,\ \mathrm{ad}^0\bar{\rho}(1))$$

holds. Here

$$h_v = \dim_{k_\lambda} H^0(F_v,\ \mathrm{ad}^0\bar{\rho}(1)) - \dim_{k_\lambda} H^1(F_v,\ \mathrm{ad}^0\bar{\rho})/H^1_{\mathrm{def}(v)}(F_v,\ \mathrm{ad}^0\bar{\rho})$$

for $v\in\Sigma_{\mathscr{D}}$, and

$$h_v = \dim_{k_\lambda} H^0(F_v,\mathrm{ad}^0\bar{\rho}(1))$$

for $v\in I_F$. Since $\bar{\rho}|_{G_{F(\zeta_\ell)}}$ is absolutely irreducible,

$$\dim_{k_\lambda} H^0(F,\ \mathrm{ad}^0\bar{\rho}) = \dim_{k_\lambda} H^0(F,\ \mathrm{ad}^0\bar{\rho}(1)) = 0.$$

Since $\bar{\rho}$ is odd for any complex conjugation,

$$\sum_{v\in I_F} h_v = 2[F:\mathbb{Q}].$$

If $v\in\Sigma_{\mathscr{D}},\ v\nmid\ell$, then $h_v=0$. In this case, the local condition is finite, and this follows from Proposition 3.5, since $\dim_{k_\lambda} H^1_f(F_v,\ \mathrm{ad}^0\bar{\rho}) = \dim_{k_\lambda} H^0(F_v,\mathrm{ad}^0\bar{\rho})$. When $v|\ell$, we show $h_v\le -2[F_v:\mathbb{Q}_\ell]$. In the nearly ordinary and nearly ordinary finite case, this follows from Theorem 3.16, and Tate's local Euler characteristic formula. In the flat case we have

$$\dim_{k_\lambda} H^1_{\mathfrak{fl}}(F_v,\ \mathrm{ad}^0\bar{\rho}|_{F_v}) = \dim_{k_\lambda} H^0(F_v,\ \mathrm{ad}^0\bar{\rho}|_{F_v}) + [F_v:\mathbb{Q}_\ell]$$

by Theorem 3.20, and the claim follows again by Tate's local Euler characteristic formula.

So the total sum can not be strictly positive.

For (2), by [51], proposition 1.6 and Lemma 9.14

$$\dim_{k_\lambda} H^1_{\mathscr{D}_Q}(F,\ \mathrm{ad}^0\bar{\rho})/H^1_{\mathscr{D}^*_Q}(F,\ \mathrm{ad}^0\bar{\rho}(1)) = \dim_{k_\lambda} H^1_{\mathscr{D}}(F,\ \mathrm{ad}^0\bar{\rho})/H^1_{\mathscr{D}^*}(F,\ \mathrm{ad}^0\bar{\rho}(1))$$
$$+\sum_{v\in Q}\dim_{k_\lambda} H^0(F_v,\ \mathrm{ad}^0\bar{\rho}(1)) \le \sharp Q$$

holds. Here we have used (1), and the fact that $\dim_k H^0(F_v,\ \mathrm{ad}^0\bar{\rho}(1)) = 1$ for each $v\in Q$, since $q_v\equiv 1 \mod \ell$, and $\bar{\rho}(\mathrm{Fr}_v)$ is a regular semi-simple element in $\mathrm{GL}_2(k_\lambda)$. □

We prove Proposition 9.13. By Lemma 9.14, (2), an element Q in $\mathscr{X}_{\mathscr{D}}$ satisfies the inequality

$$(*)\qquad \dim_k H^1_{\mathscr{D}_Q}(F,\ \mathrm{ad}^0\bar{\rho}) \le \#Q$$

if

$$(**)\qquad \dim_{k_\lambda} H^1_{\mathscr{D}^*_Q}(F,\ \mathrm{ad}^0\bar{\rho}(1)) = 0.$$

For any integer $n\ge 1$, it is clear that a finite set Q given by the following lemma satisfies conclusions (1)-(3) of Proposition 9.13 since $(*)$, and hence $(**)$, is satisfied for this Q with equality $\#Q = \dim_{k_\lambda} H^1_{\mathscr{D}^*}(F,\ \mathrm{ad}^0\bar{\rho}(1))$.

Lemma 9.15. *(cf.* [51]*, 3.8) For a prime $\ell\ge 3$, let F be a number field such that $[F(\zeta_\ell):F]$ is even. Assume that $\bar{\rho}|_{F(\zeta_\ell)}$ is absolutely irreducible. When $\ell=5$, the following case is excluded: the projective image $\bar{G}$ of $\bar{\rho}$ is isomorphic to* $\mathrm{PGL}_2(\mathbb{F}_5)$*, and the mod ℓ-cyclotomic character $\bar{\chi}_{\mathrm{cycle}}$ factors through $G_F\to\bar{G}^{\mathrm{ab}}\simeq\mathbb{Z}/2$.*

Then for any $n\ge 1$, there is an element $Q\in\mathscr{X}_{\mathscr{D}}$ which satisfies the following conditions.

(1) *For $v\in Q$, v is of degree one, and $q_v\equiv 1 \mod \ell^n$ holds.*

(2) *The restriction map*

$$H^1_{\mathscr{D}^*}(F,\ \mathrm{ad}^0\,\bar\rho(1)) \longrightarrow \bigoplus_{v\in Q} H^1_f(F_v,\ \mathrm{ad}^0\,\bar\rho(1))$$

is bijective. In particular the kernel $H^1_{\mathscr{D}^*_Q}(F,\ \mathrm{ad}^0\,\bar\rho(1))$ *vanishes.*

Proof of Lemma 9.15. Fix an integer $n \geq 1$. We apply Proposition 9.6, and Corollary 9.12. We take $\bar\chi_{\mathrm{cycle}}$ as μ. Let F be the number field in consideration, F' the n-th layer of the cyclotomic $\mathbb{Z}_\ell$-extension of F, $M = \mathrm{ad}^0\,\bar\rho(1)$, $H = \ker\mu$. Since F'/F is an ℓ-extension, $\mu|_{F'}$ has the same order as μ.

Let x be a non-zero cohomology class in $H^1_{\mathscr{D}^*}(F,\ M)$, which is not exceptional for (M, H) over F. First we show that x is not exceptional for (M, H) over F'. Since $\mathrm{Gal}(F'/F)$ is a cyclic group of ℓ-power order, $\rho|_{G_{F'}}$ remains absolutely irreducible by Lemma 9.5, (1), and hence $H^0(F',\ M) = \{0\}$, and the restriction $x_{F'}$ of x is non-zero. If x_F is exceptional, by Proposition 9.6, $\ell = 5$ and the projective image of $\bar\rho|_{G_{F'}}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$ and $\mu|_H$ factors through $\mathrm{PGL}_2(\mathbb{F}_5)^{\mathrm{ab}}$. By Lemma 9.5, (2), the projective image of ρ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, and hence x is exceptional over F.

So the condition (1) of Corollary 9.12 is satisfied. For the condition (2) of Corollary 9.12, for any irreducible subspace N, we take g_N as the image of an element σ given by Lemma 9.2 in the non-dihedral case (F'/F is a cyclic extension), and by Lemma 9.4 in the dihedral case (the restriction of $\bar\rho$ to $F'(\zeta_\ell)$ is absolutely irreducible by the assumption on $\bar\rho$ and Lemma 9.5, (1)).

Thus Corollary 9.12 is applied, and we have a degree 1 place v of F such that v splits completely in F', the image of Fr_v in G_F belongs to $G_{F'} \cap H$, the restriction of x to $H^1_f(F_v, M)$ is non-zero, and $\mathrm{ad}^0\,\bar\rho(\mathrm{Fr}_v)$, and hence $\bar\rho(\mathrm{Fr}_v)$, are regular and semi-simple. Since v splits completely in F' and $\mu(\mathrm{Fr}_v) = 1$, $q_v \equiv 1 \mod \ell^n$.

$\dim_{k_\lambda} H^1_f(F_v,\ M) = 1$ because $\bar\rho(\mathrm{Fr}_v)$ is a regular semi-simple element in $\mathrm{GL}_2(k_\lambda)$. Then the image of x under the restriction map

$$H^1_{\mathscr{D}^*}(F,\ M) \longrightarrow H^1_f(F_v,\ M)$$

spans $H^1_f(F_v,\ M)$. Continuing successively by taking a non-zero element in the kernel which is not exceptional for (M, H), we have a finite set Q which belongs to $\mathscr{X}_{\mathscr{D}}$, such that

$$H^1_{\mathscr{D}^*}(F, M) \longrightarrow \bigoplus_{v\in Q} H^1_f(F_v,\ M)$$

is surjective, and the kernel consists of classes which are exceptional for (M, H) over F. We have excluded the exceptional case, so we finish the proof.

□

10. Congruence modules

We calculate the cohomological congruence modules made from modular varieties. For elliptic modular curves and constant sheaves, the idea is due to Ribet [38]. In [51], §2 this is generally discussed, to reduce the estimate of the Selmer group to the minimal case. We proceed in the same way as in [51], assuming cohomological universal injectivity in the Shimura curve case.

10.1. **Dual construction.** For an absolutely irreducible representation $\bar{\rho}$ and a deformation type $\mathscr{D}$, we define an $R_{\mathscr{D}}$-module $\hat{M}_{\mathscr{D}} = M_{\mathscr{D}} \oplus M_{\mathscr{D}}^{\mathrm{op}}$ which contains $M_{\mathscr{D}}$ as a direct summand.

In general, for a Galois representation $\rho : G_F \to \mathrm{GL}_2(A)$, we define ρ^{op} as

$$\rho^{\mathrm{op}} = \rho^{\vee}(-1) = (\det\rho)^{-1} \cdot \rho(-1).$$

It is easily checked that $(\rho^{\mathrm{op}})^{\mathrm{op}} = \rho$, and for a place $v \nmid \ell$ where ρ is unramified,

$$\begin{cases} \mathrm{trace}\rho^{\mathrm{op}}(\mathrm{Fr}_v) = q_v \cdot (\det\rho(\mathrm{Fr}_v))^{-1} \cdot \mathrm{trace}\rho(\mathrm{Fr}_v) \\ \det\rho^{\mathrm{op}}(\mathrm{Fr}_v) = q_v^2 \cdot (\det\rho(\mathrm{Fr}_v))^{-1} \end{cases}$$

hold.

We fix a discrete infinity type (k, w). For a deformation type $\mathscr{D}$ of $\bar{\rho}$, $M_{\mathscr{D}}$ is regarded as a submodule of $H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_K, \bar{\mathscr{F}}_{(k,w)})$ as in §7.4.

$$M_{\mathscr{D}}^{\mathrm{op}} = \mathrm{Hom}_{o_{\mathscr{D}}}(M_{\mathscr{D}}, o_{\mathscr{D}})$$

is regarded as a submodule of $H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_K, \bar{\mathscr{F}}_{(k,-w)})(q_{\bar{\rho}})$ by Poincaré duality discussed in §4.3.

We define the deformation type $\mathscr{D}^{\mathrm{op}}$ for $\bar{\rho}^{\mathrm{op}}$. The deformation functions and the coefficient rings are the same for $\mathscr{D}$ and $\mathscr{D}^{\mathrm{op}}$. At $v|\ell$, the nearly ordinary type (resp. flat twist type) of $\mathscr{D}^{\mathrm{op}}$ is $(\det\bar{\rho})^{-1}_{\mathrm{lift}}|_{F_v} \cdot \kappa_{\mathscr{D},v}(-1)$, where $\kappa_{\mathscr{D},v}$ is the nearly ordinary type (resp. flat twist type) of $\mathscr{D}$.

The K-type for $\bar{\rho}^{\mathrm{op}}$ and $\mathscr{D}^{\mathrm{op}}$ at a finite place v is the same as that of $\bar{\rho}$ except for the K-character. The K-character is defined as

$$\nu_{\mathrm{def}_{\mathscr{D}^{\mathrm{op}}}}(\bar{\rho}|_{F_v}) = \nu_{\mathrm{def}_{\mathscr{D}}}(\bar{\rho}|_{F_v})^{-1}.$$

Thus we have the ℓ-adic Hecke algebra $T_{\mathscr{D}^{\mathrm{op}}}$ and $T_{\mathscr{D}^{\mathrm{op}}}$-module $M_{\mathscr{D}^{\mathrm{op}}}$ for $\bar{\rho}^{\mathrm{op}}$.

Proposition 10.1. (1) *$M_{\mathscr{D}}^{\mathrm{op}}$ is canonically isomorphic to the module associated to $\bar{\rho}^{\mathrm{op}}$ of deformation type $\mathscr{D}^{\mathrm{op}}$.*

(2) *There is an $o_{\mathscr{D}}$-algebra isomorphism $T_{\mathscr{D}} \simeq T_{\mathscr{D}^{\mathrm{op}}}$ between the ℓ-adic Hecke algebras attached to $\bar{\rho}$ and $\bar{\rho}^{\mathrm{op}}$, which maps T_v to $T_{v,v}^{-1} \cdot T_v$, $T_{v,v}$ to $T_{v,v}^{-1}$ for $v \notin \Sigma_{\mathscr{D}}$.*

Proof of Proposition 10.1. The compatibility of standard Hecke operators T_v and $T_{v,v}$ follows from Proposition 4.7. One also checks the compatibility of for $\tilde{U}(p_v)$ and $\tilde{U}(p_v, p_v)$-operators under Poincar'e duality. □

By Proposition 10.1, $\hat{M}_{\mathscr{D}} = M_{\mathscr{D}} \oplus M_{\mathscr{D}}^{\mathrm{op}}$ is regarded as a $T_{\mathscr{D}}$-module, and self-dual as a $T_{\mathscr{D}}$-module. Thus $\hat{M}_{\mathscr{D}}$ has a pairing

$$\langle\ ,\ \rangle_{\mathscr{D}} : \hat{M}_{\mathscr{D}} \times \hat{M}_{\mathscr{D}} \longrightarrow o_{\mathscr{D}}$$

which induces $\hat{M}_{\mathscr{D}} \simeq \mathrm{Hom}_{o_{\mathscr{D}}}(\hat{M}_{\mathscr{D}}, o_{\mathscr{D}})$ as $T_{\mathscr{D}}$-modules.

10.2. **Congruence modules (I).** For a division quaternion algebra D as in §4.3, we assume that D is split at all $v|\ell$. For an F-factorizable compact open subgroup K of $G_D(\mathbb{A}_{\mathbb{Q},f})$ and a finite set Σ of finite places which contains Σ_K, take $v \notin \Sigma$ such that

$$K_v = \mathrm{GL}_2(o_{F_v})^{\ell} = \text{ the inverse image of } \Delta_v \text{ by } \mathrm{GL}_2(o_{F_v}) \overset{\det}{\to} o_{F_v}^{\times} \to k(v)^{\times}.$$

Consider two degeneracy maps $\mathrm{pr}_1,\ \mathrm{pr}_2 : S_{K \cap K_0(v)} \to S_K$ defined in §5.3, and

$$\varphi_{(k,w)} : H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 2} \overset{\mathrm{pr}_1^* + \mathrm{pr}_2^*}{\longrightarrow} H^{q_D}(S_{K \cap K_0(v)},\ \bar{\mathscr{F}}_{(k,w)}).$$

is the map considered there. When $q_D = 0$, $\varphi_{(k,w)}$ is universally injective, that is, injective and the image is an o_{E_λ}-direct summand up to the modules of residual type (with respect

to the action of the convolution algebra H_Σ). If $q_D = 1$, $\varphi_{(k,w)}$ satisfies the same properties under Hypothesis 5.9.

By taking the dual of $\varphi_{(k,-w)}$ induced by Poincaré duality, with appropriate Tate twists, we have

$$\tilde{\varphi}_{(k,w)} : H^{q_D}(S_{K\cap K_0(v)},\ \bar{\mathscr{F}}_{(k,w)}) \xrightarrow{\varphi^\vee_{(k,-w)}(-q_D)} H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 2}.$$

By the construction of duality pairing, $\tilde{\varphi}_{(k,w)}$ is the map $(\mathrm{pr}_1)_! + (\mathrm{pr}_2)_!$ obtained from the trace map. The following lemma is originally due to Ribet [39].

Lemma 10.2. *Assume that the v-type is $((2,\dots,2),0)$ if $v|\ell$. Then*

$$\tilde{\varphi}_{(k,w)} \circ \varphi_{(k,w)}(H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 2}) = (U(p_v)^2 - T_{v,v})H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 2}$$

holds up to the modules of residual type (Hypothesis 5.9 is unnecessary for this statement).

Proof of Lemma 10.2. First we calculate $\tilde{\varphi}_{(k,w)} \circ \varphi_{(k,w)}$ explicitly.

$$(*) \qquad \tilde{\varphi}_{(k,w)} \circ \varphi_{(k,w)} = \begin{pmatrix} 1+q_v & T_v \\ T_{v,v}^{-1}\cdot T_v & 1+q_v \end{pmatrix}$$

Here the multiplication is on the left. $(*)$ is seen as follows. By Proposition 5.5, the localization of $H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 2}$ at some maximal ideal of H_K which is not of residual type is o_{E_λ}-free, so it suffices to check $(*)$ after tensoring with E_λ.

Take an element $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w),E_\lambda})^{\oplus 2}$. Then

$$\begin{aligned} &\tilde{\varphi}_{(k,w)} \circ \varphi_{(k,w)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= (\mathrm{pr}_1)_!\,\mathrm{pr}_1^* f_1 + (\mathrm{pr}_2)_!\,\mathrm{pr}_1^* f_1 + (\mathrm{pr}_1)_!\,\mathrm{pr}_2^* f_2 + (\mathrm{pr}_2)_!\,\mathrm{pr}_2^* f_2 \\ &= (q_v+1)f_1 + T_{v,v}^{-1} f_1 + T_v f_2 + (q_v+1)f_2. \end{aligned}$$

Here we have used that $(\mathrm{pr}_1)_!\,\mathrm{pr}_2^* = T_v$, and $(\mathrm{pr}_2)_!\,\mathrm{pr}_1^*$ is the dual correspondence $T_{v,v}^{-1}\cdot T_v$. Thus $(*)$ is shown.

Similarly, we have

$$U_v \circ \varphi_{(k,w)} = \varphi_{(k,w)} \circ \begin{pmatrix} 0 & -T_{v,v} \\ q_v & T_v \end{pmatrix},$$

$$(U_v^2 - T_{v,v}) \circ \varphi_{(k,w)} = \begin{pmatrix} -T_{v,v} & 0 \\ T_v & -T_{v,v} \end{pmatrix} \circ (\tilde{\varphi}_{(k,w)} \circ \varphi_{(k,w)}).$$

□

Consider an absolutely irreducible representation $\bar{\rho}$ with a deformation type $\mathscr{D}$. We assume that the deformation function satisfies $\mathrm{def}_{\mathscr{D}}(v) =$ **n.o.f.**. Let $\mathscr{D}'$ be the deformation type of $\bar{\rho}$ which has the same data as $\mathscr{D}$ except the deformation function $\mathrm{def}_{\mathscr{D}'}$. $\mathrm{def}_{\mathscr{D}'}(v) =$ **n.o.**, and $\mathrm{def}_{\mathscr{D}'}$ takes the same value as $\mathrm{def}_{\mathscr{D}}$ at the other places.

We have defined $T_{\mathscr{D}}$ and $T_{\mathscr{D}'}$-modules $M_{\mathscr{D}}$ and $M_{\mathscr{D}'}$.

We show that there is a canonical surjective homomorphism $\beta : T_{\mathscr{D}'} \to T_{\mathscr{D}}$ and $T_{\mathscr{D}'}$-homomorphism

$$\xi : M_{\mathscr{D}} \longrightarrow M_{\mathscr{D}'},$$

with the calculation of the congruence modules. We may assume that the v-type satisfies $(k_v, w) = ((2,\dots,2), w)$ by our assumption 6.11, and hence w is even. We may twist $\bar{\mathscr{F}}_{(k,w)}$ by $\mathrm{N}_{D/F}^{\frac{w}{2}}$, and assume that $w = 0$.

We use the same notation as in §7.4. In particular we make a choice of an auxiliary place y and a set of finite places S to make compact open subgroups in consideration small.

For $K = \ker \nu_{\mathscr{D}_y}|_{\tilde{K}_{\mathscr{D}_y}}$, there is a canonical map

$$(\dagger_1) \qquad H^{q_{\bar\rho}}_{\text{stack}}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 2} \longrightarrow H^{q_{\bar\rho}}_{\text{stack}}(S_{(K_v \cap K_0(m_v))\cdot K^v},\ \bar{\mathscr{F}}_{(k,w)})$$

induced by the degeneracy maps. For $\Sigma' = \Sigma_{\mathscr{D}'_y} \cup S = \Sigma_{\mathscr{D}_y} \cup S \cup \{v\}$, let $\tilde{m}_{\Sigma',\bar\rho}$ be the maximal ideal of $H_{K^{\Sigma'}} = H(G_D(\mathbb{A}_{\mathbb{Q},f}), K^{\Sigma'})_{o_{\mathscr{D}}}$ corresponding to $\bar\rho$.

We localize $(\dagger_1)$ at $\tilde{m}_{\Sigma',\bar\rho}$, and take the part where $\tilde{K}_{\mathscr{D}'_y}$ acts via $\nu_{\mathscr{D}'_y}$. Thus we have an injective $o_{\mathscr{D}}$-homomorphism

$$(\dagger_2) \qquad (\tilde{M}^y_{\mathscr{D}_y})^{\oplus 2} \hookrightarrow \tilde{M}^y_{\mathscr{D}'_y}.$$

Let $\tilde{T}^{\text{ß}}_{\mathscr{D}}$ (resp. $\tilde{T}^{\text{ß}}_{\mathscr{D}'}$) be the image of H_{K^Σ} in $\operatorname{End}_{o_{\mathscr{D}}} \tilde{M}^y_{\mathscr{D}_y}$ (resp. $\operatorname{End}_{o_{\mathscr{D}}} \tilde{M}^y_{\mathscr{D}'_y}$). In the case of **n.o.f.**, $\tilde{U}(p_v) = U(p_v)$-operator on the right hand side of $(\dagger_2)$ satisfies the relation

$$U^2 - T_v \cdot U + q_v T_{v,v} = 0,$$

over $\tilde{T}^{\text{ß}}_{\mathscr{D}}$, since $U(p_v)$ is given by

$$\begin{pmatrix} T_v & -T_{v,v} \\ q_v & 0 \end{pmatrix}$$

as in Lemma 10.2. Let $A = \tilde{T}^{\text{ß}}_{\mathscr{D}}[U]/(U^2 - T_v \cdot U + q_v T_{v,v})$. A acts faithfully on $(\tilde{M}^y_{\mathscr{D}_y})^{\oplus 2}$, where U acts as $U(p_v)$. Then $(\dagger_2)$ is compatible with the action of Hecke algebras (including $U(p_v)$-operator), we have a canonical surjective homomorphism

$$\tilde{T}^{\text{ß}}_{\mathscr{D}'}[U(p_v)] \twoheadrightarrow A.$$

Since $\operatorname{def}_{\mathscr{D}'}(v) = \mathbf{n.o.}$, there is only one non-zero root of $(\dagger_3)$ mod $\tilde{m}^{\text{ß}}_{\mathscr{D}_y}$, which defines a maximal ideal m_A of A. The localization A_{m_A} is isomorphic to $T^{\text{ß}}_{\mathscr{D}}$, and $((\tilde{M}^y_{\mathscr{D}_y})^{\oplus 2})_{m_A} \simeq \tilde{M}^y_{\mathscr{D}_y}$.

By Theorem 7.18, there is a maximal ideal ideal $\tilde{m}'$ of $\tilde{T}^{\text{ß}}_{\mathscr{D}'}[U(p_v)]$, and the localization $(\tilde{T}^{\text{ß}}_{\mathscr{D}'}[U(p_v)])_{\tilde{m}'}$ is equal to $T_{\mathscr{D}'}$. Evidently $\tilde{m}'$ is above m_A since the condtion at v is **n.o.** or **n.o.f.**, and $A_{\tilde{m}'A} \simeq T_{\mathscr{D}_y}$.

Thus we have

$$\beta : T_{\mathscr{D}'} \xrightarrow{\sim} T_{\mathscr{D}'_y} \longrightarrow T_{\mathscr{D}_y} \xrightarrow{\sim} T_{\mathscr{D}},$$

using Proposition 7.12. Moreover,

$$\xi : M_{\mathscr{D}} \longrightarrow M_{\mathscr{D}'}$$

is induced by localization at $\tilde{m}'$. By applying this result to $\bar\rho^{\text{op}}$ and $\mathscr{D}^{\text{op}}$, we have a $T_{\mathscr{D}'}$-homomorphism

$$\hat\xi : \hat{M}_{\mathscr{D}} \longrightarrow \hat{M}_{\mathscr{D}'},$$

and we view the dual $(\hat\xi)^\vee : \hat{M}_{\mathscr{D}'} \to \hat{M}_{\mathscr{D}}$ also as a $T_{\mathscr{D}'}$-homomorphism. The restriction of $(\hat\xi)^\vee$ to $M_{\mathscr{D}}$ is $\xi^\vee$ by the construction.

Proposition 10.3. *Let $\bar\rho$ be an absolutely irreducible representation with a deformation type $\mathscr{D}$ which satisfies $\operatorname{def}_{\mathscr{D}}(v) = \mathbf{n.o.f.}$ at $v|\ell$. Let $\mathscr{D}'$ be the deformation type of $\bar\rho$ which has the same data as $\mathscr{D}$, and the deformation function $\operatorname{def}_{\mathscr{D}'}$ satisfies $\operatorname{def}_{\mathscr{D}'}(v) = \mathbf{n.o.}$, and takes the same value as $\operatorname{def}_{\mathscr{D}}$ other than v.*

(1) *There is a surjective homomorphism $\beta : T_{\mathscr{D}'} \to T_{\mathscr{D}}$, and for the canonical map $\xi : M_{\mathscr{D}} \to M_{\mathscr{D}'}$,*

$$\xi^\vee \circ \xi(M_{\mathscr{D}}) = \Delta \cdot M_{\mathscr{D}}, \quad \Delta = \tilde{U}(p_v)^2 - \tilde{U}(p_v, p_v).$$

Here $\xi^\vee : M_{\mathscr{D}'} \to M_{\mathscr{D}}$ is the map defined by self-duality pairings $\langle\ ,\ \rangle_{\mathscr{D}}$ and $\langle\ ,\ \rangle_{\mathscr{D}'}$.

(2) *Similarly, we have*

$$(\hat{\xi})^\vee \circ \hat{\xi}(\hat{M}_{\mathscr{D}}) = \Delta \cdot \hat{M}_{\mathscr{D}}.$$

Here Δ is the same as in (1).

(3) *Assume Hypothesis 5.9 if $q_{\bar{\rho}} = 1$. Then $\hat{\xi}(\hat{M}_{\mathscr{D}})$ is an $o_{\mathscr{D}}$-direct summand of $\hat{M}_{\mathscr{D}'}$.*

Proof of Proposition 10.3. (1) follows from Lemma 10.2, since $\xi^\vee|_{M_{\mathscr{D}'}}$ defined by the self-duality of $\hat{M}_{\mathscr{D}'}$ is equal to the map deduced from $\tilde{\varphi}_{(k,w)}$. For (2), we should check that the element $\tilde{U}(p_v)^2 - \tilde{U}(p_v,p_v)$ in $T_{\mathscr{D}^{\mathrm{op}}}$ generate the same ideal as generated by $\tilde{U}(p_v)^2 - \tilde{U}(p_v,p_v)$ in $T_{\mathscr{D}}$ by the isomorphism $T_{\mathscr{D}} \simeq T_{\mathscr{D}^{\mathrm{op}}}$. This follows from the equality $\tilde{U}(p_v)^2 - \tilde{U}(p_v,p_v) = \tilde{U}(p_v,p_v)^2((\tilde{U}(p_v,p_v)^{-1}\tilde{U}(p_v))^2 - \tilde{U}(p_v,p_v)^{-1})$.

(3) is clear from the definition of ξ. □

10.3. **Congruence modules (II).** From now on, we assume that the deformation type $\mathscr{D}$ of $\bar{\rho}$ satisfies $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{f}$. Let $\mathscr{D}'$ be the deformation type of $\bar{\rho}$ which has the same data as $\mathscr{D}$ except for the deformation function $\mathrm{def}_{\mathscr{D}'}$. $\mathrm{def}_{\mathscr{D}'}(v) = \mathbf{u}$, and $\mathrm{def}_{\mathscr{D}'}$ takes the same value as $\mathrm{def}_{\mathscr{D}}$ at the other places.

In this subsection, we assume moreover that $\bar{\rho}$ is either of type 1_{SP} or 1_{PR} at v.

Let D be a division algebra with $q_D \leq 1$, K an F-factorizable compact open subgroup of $G_D(\mathbb{A}_{\mathbb{Q},f})$, and for a place $v \nmid \ell$ and an integer $c \geq 1$, we assume that $K_v = K_1(m_{F_v}^c) \cap \mathrm{GL}_2(o_{F_v})^\ell$ in the notation of §10.2. The degeneracy maps define

$$\xi_{1,(k,w)} : H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 2} \xrightarrow{\mathrm{pr}_1^* + \mathrm{pr}_2^*} H^{q_D}(S_{K\cap K_0(v^{c+1})},\ \bar{\mathscr{F}}_{(k,w)}).$$

Note that $\xi_{1,(k,w)}$ may not be injective, even on the part which is not of residual type. $\xi_{1,(k,w)}^\vee$ is defined similarly as in §10.2. Then, as in the proof of Lemma 10.2, we have

$$(*) \qquad (\xi_{1,(k,w)})^\vee \circ \xi_{1,(k,w)} = \begin{pmatrix} q_v & U(p_v) \\ U(p_v,p_v)^{-1} \cdot U(p_v) & q_v \end{pmatrix}.$$

Let $\tau_{(k,w)} : H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)}) \to H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 2}$ be the map defined by

$$\tau_{(k,w)} : c \longmapsto \begin{pmatrix} -q_v \\ U(p_v,p_v)^{-1} \cdot U(p_v) \end{pmatrix} \cdot c.$$

By $(*)$,

$$\begin{aligned} \tau_{(k,-w)}^\vee \circ \xi_{1,(k,w)}^\vee \circ \xi_{1,(k,w)} \circ \tau_{(k,w)} &= \begin{pmatrix} -q_v & U(p_v) \end{pmatrix} \cdot \begin{pmatrix} U(p_v,p_v)^{-1} \cdot U(p_v)^2 - q_v^2 \\ 0 \end{pmatrix} \\ &= -q_v(U(p_v,p_v)^{-1} \cdot U(p_v)^2 - q_v^2). \end{aligned}$$

Since we allow the unrestricted condition on the determinant when $\mathrm{def}_{\mathscr{D}'} = \mathbf{u}$, we need to discuss the passage from $K_0(m_v^{c+1}) \cap GL_2(o_{F_v})^\ell$ to $K_1(m_v^{c+1})$.

We define $\Delta_{K,v}$ as the quotient of Δ_v by the image of $Z(F) \cap K$. Then

$$\pi : S_{K_1(m_{F_v}^{c+1})\cdot K^v} \longrightarrow S_{K\cap K_0(v^{c+1})}$$

is a $\Delta_{K,v}$-torsor, and the composition

$$H^{q_D}(S_{K\cap K_0(v^{c+1})}, \bar{\mathscr{F}}_{(k,w)}) \xrightarrow{\pi^*} H^{q_D}(S_{K_1(m_{F_v}^{c+1})\cdot K^v}, \bar{\mathscr{F}}_{(k,w)}) \xrightarrow{\pi_!} H^{q_D}(S_{K\cap K_0(v^{c+1})}, \bar{\mathscr{F}}_{(k,w)})$$

induced by the trace map is the multiplication by $\sharp\Delta_{K,v}$.

Thus we have

Lemma 10.4. *Assume that $v \nmid \ell$.*

(1) *We obtain*

$$\tau^{\vee}_{(k,-w)}\circ\xi^{\vee}_{1,(k,w)}\circ\xi_{1,(k,w)}\circ\tau_{(k,w)}(H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)}))=(U(p_v,p_v)^{-1}\cdot U(p_v)^2-q_v^2)H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})$$

up to the modules of residual type.

(2) *The equality*

$$\begin{aligned}&\tau^{\vee}_{(k,-w)}\circ\xi^{\vee}_{1,(k,w)}\circ\pi_!\circ\pi^*\circ\xi_{1,(k,w)}\circ\tau_{(k,w)}(H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)}))\\ &\quad=\sharp\Delta_{K,v}\cdot(U(p_v,p_v)^{-1}\cdot U(p_v)^2-q_v^2)H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})\end{aligned}$$

holds up to the modules of residual type.

We apply the result to $K=\ker\nu_{\mathscr{D}_y}|_{\tilde{K}_{\mathscr{D}_y}}$. Recall that we assume that $\bar{\rho}$ is of type 1_{SP} or 1_{PR}. Let $c=c(\bar{\rho})$ be the integer ≥ 1 defined in §3.6. To define $\beta: T_{\mathscr{D}'}\to T_{\mathscr{D}}$ in this case,

$$A=\tilde{T}^{\beta}_{\mathscr{D}}[U(p_v)][U]/(U(U-U(p_v))).$$

The A-action on $H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 2}$ is given by

$$U\longmapsto\begin{pmatrix}U(p_v) & q_v\\ 0 & 0\end{pmatrix}.$$

A direct calculation in this case shows that this U-action is compatible with the $U(p_v)$-action on $H^{q_{\bar{\rho}}}(S_{K_1(m_{F_v}^{c+1})\cdot K^v},\ \bar{\mathscr{F}}_{(k,w)})$, and A is regarded as a $\tilde{T}^{\beta}_{\mathscr{D}'}[U(p_v)]$-algebra.

By the same argument as in the case of **n.o.f.**, there is a maximal ideal $\tilde{m}'$ of $\tilde{T}^{\beta}_{\mathscr{D}'}[U(p_v)]$, and the localization $(\tilde{T}^{\beta}_{\mathscr{D}'}[U(p_v)])_{\tilde{m}'}$ is equal to $T_{\mathscr{D}'}$. Since $\mathrm{def}_{\mathscr{D}'}(v)=\mathbf{u}$, $m_A=\tilde{m}'A$ contains U, and the localization A_{m_A} at m_A is isomorphic to $T_{\mathscr{D}_y}$ (though we have omitted $U(p_v)$-operator in the definition of $T_{\mathscr{D}_y}$, by Proposition 7.26 it is recovered from $\rho^{\mathrm{mod}}_{\mathscr{D}}$). Thus we have a surjective homomorphism $\beta: T_{\mathscr{D}'}\to T_{\mathscr{D}}$.

Then $\xi_{1,(k,w)}$ becomes injective on the localization by m_S by the cohomological universal exactness (Proposition 5.13 with $n=c$), since there is no contribution from $H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_{(\mathrm{GL}_2(o_{F_v})^{\ell}\cap K_1(m_{F_v}^{c-1})\cdot K^v},\ \bar{\mathscr{F}}_{(k,w)})$ as $c=c(\bar{\rho})$ is the conductor of G_{F_v}-representation $\bar{\rho}|_{F_v}\otimes\bar{\kappa}_v^{-1}$ and the level can not be smaller. It also follows that the image of $\xi_{1,(k,w)}$ is an o_{λ}-direct summand after localization. After localization at m_A, $\tau_{(k,w)}$ is an isomorphism, and π is universally injective by Proposition 5.8.

Thus we have a $T_{\mathscr{D}'}$-homomorphism

$$\xi=(\pi\circ\xi_{1,(k,w)}\circ\tau_{(k,w)})_{m_A}: M_{\mathscr{D}}\longrightarrow M_{\mathscr{D}'}.$$

As for the calculation of the congruence module, we treat the 1_{SP}-case first. Since all representations which contributes to $T_{\mathscr{D}}$ are special representations at v, $U(p_v)^2=U(p_v,p_v)$ holds in A_{m_A}. By Lemma 10.4, we have

Proposition 10.5. *Assume that $\bar{\rho}$ is of type 1_{SP} at v, and $\mathrm{def}_{\mathscr{D}}(v)=\mathbf{f}$.*

(1) *There is a surjective homomorphism $\beta: T_{\mathscr{D}'}\to T_{\mathscr{D}}$, and for the canonical map $\xi: M_{\mathscr{D}}\to M_{\mathscr{D}'}$,*

$$\xi^{\vee}\circ\xi(M_{\mathscr{D}})=\Delta\cdot M_{\mathscr{D}},\quad \Delta=\sharp(\ker(C_{F,P^{\mathbf{u}}_{\mathscr{D}'},\ell}\to C_{F,P^{\mathbf{u}}_{\mathscr{D}},\ell}))\cdot(q_v^2-1).$$

Here, $\xi^{\vee}: M_{\mathscr{D}'}\to M_{\mathscr{D}}$ is the map defined by Poincaré duality.

(2)

$$(\hat{\xi})^{\vee}\circ\hat{\xi}(\hat{M}_{\mathscr{D}})=\Delta\cdot\hat{M}_{\mathscr{D}}.$$

Here, Δ is the same as in (1).

(3) *$\hat{\xi}(\hat{M}_{\mathscr{D}})$ is an $o_{\mathscr{D}}$-direct summand of $\hat{M}_{\mathscr{D}'}$.*

Next we treat the 1_{PR}-case. Since

$$U(p_v)^2 = q_v \cdot U(p_v, p_v)$$

holds in A_{m_A}, by Lemma 10.4, we obtain

Proposition 10.6. *Assume that $\bar{\rho}$ is of type 1_{PR} at v, and $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{f}$.*

(1) *There is a surjective homomorphism $\beta : T_{\mathscr{D}'} \to T_{\mathscr{D}}$, and for the canonical map $\xi : M_{\mathscr{D}} \to M_{\mathscr{D}'}$,*

$$\xi^\vee \circ \xi(M_{\mathscr{D}}) = \Delta \cdot M_{\mathscr{D}}, \quad \Delta = \sharp(\ker(C_{F,P^{\mathbf{u}}_{\mathscr{D}'},\ell} \to C_{F,P^{\mathbf{u}}_{\mathscr{D}},\ell})) \cdot (q_v - 1).$$

Here, $\xi^\vee : M_{\mathscr{D}'} \to M_{\mathscr{D}}$ is the map defined by Poincaré duality.

(2)

$$(\hat{\xi})^\vee \circ \hat{\xi}(\hat{M}_{\mathscr{D}}) = \Delta \cdot \hat{M}_{\mathscr{D}}.$$

Here, Δ is the same as in (1).

(3) *$\hat{\xi}(\hat{M}_{\mathscr{D}})$ is an $o_{\mathscr{D}}$-direct summand of $\hat{M}_{\mathscr{D}'}$.*

10.4. **Congruence modules (III).** In this subsection, we treat the 2_{PR}-case.

For a division quaternion algebra D with $q_D \leq 1$, assume that K is an F-factorizable small compact open subgroup of $G_D(\mathbb{A}_{\mathbb{Q},f})$, and $K_v = \mathrm{GL}_2(o_{F_v})^\ell$ in the notation of §10.2 at $v \nmid \ell$. We denote $\mathrm{pr}_1,\ \mathrm{pr}_2 : S_{K\cap K_0(v)} \to S_K$ the degeneracy maps defined in §5.1. Here pr_1 corresponds to the standard inclusion of the groups. $\mathrm{pr}_3,\ \mathrm{pr}_4 : S_{K\cap K_0(v^2)} \to S_{K\cap K_0(v^2)}$ are two degeneracy maps defined similarly.

By Proposition 5.13 with $n = 1$,

$$0 \longrightarrow H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)}) \overset{(\mathrm{pr}_2^*,\ -\mathrm{pr}_1^*)}{\longrightarrow} H^{q_D}(S_{K\cap K_0(v)},\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 2} \overset{(\mathrm{pr}_3^*,\ \mathrm{pr}_4^*)}{\longrightarrow} H^{q_D}(S_{K\cap K_0(v^2)},\ \bar{\mathscr{F}}_{(k,w)})$$

is universally exact up to the modules of residual type (in Proposition 5.13, it is stated for K_{11}-structure, but the above cohomology groups with K_0-structure is obtained by taking the invariants of a subgroup of $k(v)^\times \times k(v)^\times$ by using Lemma 5.4). By using Theorem 5.10 when $q_{\bar{\rho}} = 0$, and Hypothesis 5.9 when $q_{\bar{\rho}} = 1$, the homomorphism

$$H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 3} \overset{\xi_{2,(k,w)}}{\longrightarrow} H^{q_D}(S_{K\cap K_0(v^2)},\ \bar{\mathscr{F}}_{(k,w)})$$

is universally injective up to the modules of residual type. Here $\xi_{2,(k,w)} = (\mathrm{pr}_1 \circ \mathrm{pr}_3)^* \times (\mathrm{pr}_2 \circ \mathrm{pr}_3)^* \times (\mathrm{pr}_2 \circ \mathrm{pr}_4)^*$. We define $\xi^\vee_{2,(k,w)}$ similarly as in §10.2.

The morphism

$$\pi : S_{K_1(m^2_{F_v})\cdot K^v} \longrightarrow S_{K\cap K_0(v^2)}$$

defined in §10.2 is a $\Delta_{K,v}$-torsor, and the composition

$$H^{q_D}(S_{K\cap K_0(v^2)}, \bar{\mathscr{F}}_{(k,w)}) \overset{\pi^*}{\to} H^{q_D}(S_{K_1(m^2_{F_v})\cdot K^v}, \bar{\mathscr{F}}_{(k,w)}) \overset{\pi_!}{\to} H^{q_D}(S_{K\cap K_0(v^2)}, \bar{\mathscr{F}}_{(k,w)})$$

induced by the trace map is the multiplication by $\sharp\Delta_{K,v}$. Thus we have

Lemma 10.7. *Assume that $v \nmid \ell$.*

(1) *We obtain*

$$\xi^\vee_{2,(k,w)} \circ \xi_{2,(k,w)}\ (H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 3}) = (q_v - 1)(T_v^2 - T_{v,v}(1 + q_v)^2)H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 3}$$

up to the modules of residual type.

(2) *The equality*

$$\xi^\vee_{2,(k,w)} \circ \pi_! \circ \pi^* \circ \xi_{2,(k,w)}\ (H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 3}) = \sharp\Delta_{K,v} \cdot (q_v - 1)(T_v^2 - T_{v,v}(1+q_v)^2)H^{q_D}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 3}$$

holds up to the modules of residual type.

This is shown as in [51], proposition 2.6 by 3×3-matrix calculation. So we omit the details.

To define $\beta : T_{\mathscr{D}'} \to T_{\mathscr{D}}$, we discuss as in §10.2, by taking an auxiliary place y and a set S. $K = \ker \nu_{\mathscr{D}_y}|_{\tilde{K}_{\mathscr{D}_y}}$. For the homomorphism $\tilde{\xi}$ and $\tilde{T}^{\mathrm{ß}}_{\mathscr{D}}$-algebra A defined by

$$\tilde{\xi} = \pi^* \circ \xi_{2,(k,w)} : H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 3} \xrightarrow{\xi_{2,(k,w)}} H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_{K \cap K_0(v^2)},\ \bar{\mathscr{F}}_{(k,w)})$$
$$\xrightarrow{\pi^*} H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_{K \cap K_1(v^2)},\ \bar{\mathscr{F}}_{(k,w)})$$

and

$$A = \tilde{T}^{\mathrm{ß}}_{\mathscr{D}}[U]/(U(U^2 - T_v U + q_v T_{v,v})),$$

we give the A-action on $H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 3}$ by

$$\begin{pmatrix} T_v & -T_{v,v} & 0 \\ q_v & 0 & 0 \\ 0 & q_v & 0 \end{pmatrix}.$$

One checks that the action of U is the restriction of $U(p_v)$-action on $H^{q_{\bar{\rho}}}_{\mathrm{stack}}(S_K,\ \bar{\mathscr{F}}_{(k,w)})^{\oplus 3}$. As in §10.2, there is a maximal ideal $\tilde{m}'$ of $\tilde{T}^{\mathrm{ß}}_{\mathscr{D}'}[U(p_v)]$, and the localization $(\tilde{T}^{\mathrm{ß}}_{\mathscr{D}'}[U(p_v)])_{\tilde{m}'}$ is equal to $T_{\mathscr{D}'}$ by using Proposition 7.26 and Theorem 7.18.

Since $\mathrm{def}_{\mathscr{D}'}(v) = \mathbf{u}$, $m_A = \tilde{m}' A$ contains U, and the localization A_{m_A} at m_A is isomorphic to $T_{\mathscr{D}}$. Thus we have $\beta : T_{\mathscr{D}'} \to T_{\mathscr{D}}$, and a $T_{\mathscr{D}'}$-homomorphism $\xi : M_{\mathscr{D}} \to M_{\mathscr{D}'}$ by localizing $\tilde{\xi}$.

Proposition 10.8. *Assume that $\bar{\rho}$ is of type 2_{PR} at v, and $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{f}$.*

(1) *There is a surjective homomorphism $\beta : T_{\mathscr{D}'} \to T_{\mathscr{D}}$, and for the canonical map $\xi : M_{\mathscr{D}} \to M_{\mathscr{D}'}$,*
$$\xi^\vee \circ \xi(M_{\mathscr{D}}) = \Delta \cdot M_{\mathscr{D}},$$
$$\Delta = \sharp(\ker(C_{F,P^{\mathbf{u}}_{\mathscr{D}'},\ell} \to C_{F,P^{\mathbf{u}}_{\mathscr{D}},\ell})) \cdot (q_v - 1) \cdot (T_v^2 - T_{v,v}(1+q_v)^2).$$
Here $\xi^\vee : M_{\mathscr{D}'} \to M_{\mathscr{D}}$ is the map defined by Poincaré duality.

(2) *The equality*
$$(\hat{\xi})^\vee \circ \hat{\xi}(\hat{M}_{\mathscr{D}}) = \Delta \cdot \hat{M}_{\mathscr{D}}$$
holds. Here, Δ is the same as in (1).

(3) *$\hat{\xi}(\hat{M}_{\mathscr{D}})$ is an $o_{\mathscr{D}}$-direct summand of $\hat{M}_{\mathscr{D}'}$ under Hypothesis 5.9 if $q_{\bar{\rho}} = 1$.*

10.5. **Congruence modules (IV).** Finally, we treat the cases of 0_{NE} and 0_E.

In the case of 0_{NE}, for $K = \ker \nu_{\mathscr{D}_y}|_{\tilde{K}_{\mathscr{D}_y}}$ and $c = c(\bar{\rho}) = \mathrm{Art}\bar{\rho}|_{F_v}$, we consider the morphism

$$\pi : S_{K_1(m^c_{F_v}) \cdot K^v} \longrightarrow S_{K \cap K_0(v^c)}$$

defined in §10.2 is a $\Delta_{K,v}$-torsor, and the composition

$$H^{q_D}_{\mathrm{stack}}(S_{K \cap K_0(v^c)}, \bar{\mathscr{F}}_{(k,w)}) \xrightarrow{\pi^*} H^{q_D}_{\mathrm{stack}}(S_{K_1(m^c_{F_v}) \cdot K^v}, \bar{\mathscr{F}}_{(k,w)}) \xrightarrow{\pi_!} H^{q_D}_{\mathrm{stack}}(S_{K \cap K_0(v^c)}, \bar{\mathscr{F}}_{(k,w)})$$

induced by the trace map is the multiplication by $\sharp\Delta_{K,v}$ in the notation of §10.3.

In the case of 0_E, for $K = \ker \nu_{\mathscr{D}_y}|_{\tilde{K}_{\mathscr{D}'_y}}$ and $K' = \ker \nu_{\mathscr{D}_y}|_{\tilde{K}_{\mathscr{D}_y}}$,

$$\pi : S_{K'} \longrightarrow S_K$$

is a torsor under $\tilde{\Delta}_v$. Here $\tilde{\Delta}_v$ is the ℓ-Sylow subgroup of $\mathbb{F}_{q_v^2} \simeq o_{D_{F_v}}/m_{D_{F_v}}$ (since $q_v \equiv -1 \mod \ell$ and $\ell \geq 3$, $\Delta_{K,v}$ is trivial). The composition

$$H^{q_D}_{\mathrm{stack}}(S_K, \bar{\mathscr{F}}_{(k,w)}) \xrightarrow{\pi^*} H^{q_D}_{\mathrm{stack}}(S_{K'}, \bar{\mathscr{F}}_{(k,w)}) \xrightarrow{\pi_!} H^{q_D}_{\mathrm{stack}}(S_K, \bar{\mathscr{F}}_{(k,w)})$$

induced by the trace map is the multiplication by $\sharp\tilde{\Delta}_v$.

Thus we have

Proposition 10.9. *Assume that $\bar{\rho}$ is either of type 0_E or type 0_{NE} at v, and $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{f}$.*

(1) *There is a surjective homomorphism $\beta : T_{\mathscr{D}'} \to T_{\mathscr{D}}$, and*

$$\xi^\vee \circ \xi(M_{\mathscr{D}}) = \Delta \cdot M_{\mathscr{D}}.$$

Here ξ the canonical map $M_{\mathscr{D}} \to M_{\mathscr{D}'}$, $\Delta = \sharp(\ker(C_{F,P^{\mathbf{u}}_{\mathscr{D}'},\ell} \to C_{F,P^{\mathbf{u}}_{\mathscr{D}},\ell}))$ (resp. $\sharp\tilde{\Delta}_v$) if $\bar{\rho}$ is of type 0_{NE} (resp. 0_E) at v.

(2) *The equality*

$$(\hat{\xi})^\vee \circ \hat{\xi}(\hat{M}_{\mathscr{D}}) = \Delta \cdot \hat{M}_{\mathscr{D}}$$

holds. Here, Δ is the same as in (1).

The analysis of 0_E-case was treated in [13], and by Diamond [9] with applications to deformation rings in the case where $F = \mathbb{Q}$.

11. The main theorem

In this section, we generalize the results of [51] and [9].

Theorem 11.1. *($R = T$ theorem) Let F be a totally real number field of degree d, $\bar{\rho} : G_F \to \mathrm{GL}_2(k)$ an absolutely irreducible mod ℓ-representation. We fix a deformation type $\mathscr{D}$, and assume the following conditions.*

(1) *$\ell \geq 3$, and $\bar{\rho}|_{F(\zeta_\ell)}$ is absolutely irreducible. When $\ell = 5$, the following case is excluded: the projective image $\bar{G}$ of $\bar{\rho}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, and the mod ℓ-cyclotomic character $\bar{\chi}_{\mathrm{cycle}}$ factors through $G_F \to \bar{G}^{\mathrm{ab}} \simeq \mathbb{Z}/2$ (in particular $[F(\zeta_5) : F] = 2$).*

(2) *For $v|\ell$, the deformation condition for $\bar{\rho}|_{G_{F_v}}$ is either nearly ordinary or flat. When the condition is nearly ordinary (resp. flat) at v, we assume that $\bar{\rho}|_{G_{F_v}}$ is G_{F_v}-distinguished (resp. F_v is absolutely unramified).*

(3) *There is a minimal modular lifting π of $\bar{\rho}$ in Definition 6.11.*

(4) *Hypothesis 6.7 is satisfied.*

(5) *If $\mathscr{D}$ is not minimal, we assume Hypothesis 5.9 when $q_{\bar{\rho}} = 1$.*

Then the universal deformation ring $R_{\mathscr{D}}$ of $\bar{\rho}$ of type $\mathscr{D}$ is of relative complete intersection of dimension zero over $o_{\mathscr{D}}$, and is isomorphic to the Hecke algebra $T_{\mathscr{D}}$.

We prove the following theorem at the same time.

Theorem 11.2. *(Freeness theorem) Under the same assumptions as Theorem 11.1, $M_{\mathscr{D}}$ and $\hat{M}_{\mathscr{D}}$ are free $R_{\mathscr{D}}$-modules.*

Proof of Theorem 11.1 and 11.2. Two theorems are already shown in §9 when $\mathscr{D}$ is minimal by using the Taylor-Wiles system constructed in §8. We make the reduction to the minimal case by the level-raising formalism developped in §2.

For a given deformation type $\mathscr{D}$, assume that $\pi_{\mathscr{D}} : R_{\mathscr{D}} \simeq T_{\mathscr{D}}$, $T_{\mathscr{D}}$ is a local complete intersection, and $\hat{M}_{\mathscr{D}}$ is $T_{\mathscr{D}}$-free. In the terminology of Definition 2.7, the admissible quintet $(R_{\mathscr{D}},\ T_{\mathscr{D}}, \pi_{\mathscr{D}},\ \hat{M}_{\mathscr{D}}, \langle\ ,\ \rangle_{\mathscr{D}})$ is distinguished.

Take a finite place v, and assume that $\mathrm{def}_{\mathscr{D}}(v) \in \{\mathbf{n.o.f.},\ \mathbf{f}\}$. In case of $\mathbf{n.o.f.}$, we twist by $\chi_{\mathrm{cycle}}^{-\frac{w}{2}}$, and assume that $w = 0$.

Let $\mathscr{D}'$ be the deformation type of $\bar{\rho}$ which has the same data as $\mathscr{D}$ except the deformation function $\mathrm{def}_{\mathscr{D}'}$. $\mathrm{def}_{\mathscr{D}'}(v) > \mathrm{def}_{\mathscr{D}}(v)$, and $\mathrm{def}_{\mathscr{D}'}$ takes the same value as $\mathrm{def}_{\mathscr{D}}$ at the other places.

We take a cuspidal representation π corresponding to a component of $T_{\mathscr{D}}$. By extending $E_{\mathscr{D}}$ by taking an extension of scalars if necessary, we may assume that π_f is defined over $E_{\mathscr{D}}$. π defines an $o_{\mathscr{D}}$-homomorphism $f : T_{\mathscr{D}} \to o_{\mathscr{D}}$, and a G_F-representation $\rho : G_F \to \mathrm{GL}_2(o_{\mathscr{D}})$ of type $\mathscr{D}$.

In §10, we have defined an admissible morphism

$$(R_{\mathscr{D}'},\ T_{\mathscr{D}'}, \pi_{\mathscr{D}'},\ \hat{M}_{\mathscr{D}'}, \langle\ ,\ \rangle_{\mathscr{D}'}) \longrightarrow (R_{\mathscr{D}},\ T_{\mathscr{D}}, \pi_{\mathscr{D}},\ \hat{M}_{\mathscr{D}}, \langle\ ,\ \rangle_{\mathscr{D}})$$

in the sense of Definition 2.7 under Hypothesis 5.9. The morphism commutes with scalar extension $o_{\mathscr{D}} \to o_\lambda$, and the property of being distinguished is preserved under the scalar extension.

We apply Theorem 2.8. First we check that Δ given in §10 is a non-zero divisor. In the cases of 1_{SP} 1_{PR}, 0_E and 0_{NE}, this is clear. We consider the **n.o.f**. and 2_{PR}-cases. Let π be a cuspidal representation of $\mathrm{GL}_{2,F}$ appearing in the component of $T_{\mathscr{D}}$. Then π_v belongs to principal series by the definition of $T_{\mathscr{D}}$. Let $\chi_{1,v}$ and $\chi_{2,v}$ be two quasi-characters such that π_v is isomorphic to $\pi(\chi_{1,v}, \chi_{2,v})$. We define $\alpha_v = \chi_{1,v}(p_v)$, $\beta_v = \chi_{2,v}(p_v)$ for a uniformizer p_v of F_v. By the next lemma, $\alpha_v \neq q_v^{\pm 1}\beta_v$ for any π, and hence Δ is a non-zero divisor.

Lemma 11.3. *For any cuspidal representation π of $\mathrm{GL}_{2,F}$ of a totally real field F with the infinity type (k, w), if π_v belongs to principal series, $\alpha_v \neq q_v^{\pm 1}\beta_v$ for the pair $(\alpha_v,\ \beta_v)$ defined as above.*

Proof of Lemma 11.3. Let $\pi_{\mathbf{u}} = \pi \otimes |\cdot|^{\frac{w+1}{2}}$. $\pi_{\mathbf{u}}$ is a unitary cuspidal representation of $\mathrm{GL}_{2,F}$ by our normalization. By taking the base change lift $\pi_{\mathbf{u},F'} = \mathrm{BC}(\pi_{\mathbf{u}})$ of $\pi_{\mathbf{u}}$ with respect to some abelian extension F'/F [30], one may assume that $\pi_{\mathbf{u},F'}$ has a spherical component $(\pi_{\mathbf{u},F'})_{v'}$ at a place $v'|v$. So it suffices to see $\alpha_{v'} \neq q_{v'}^{\pm 1}\beta_{v'}$ for the Satake parameter of $(\pi_{\mathbf{u},F'})_{v'}$. Since $(\pi_{\mathbf{u},F'})_{v'}$ is unitary, by [28], 2.5, Corollary, $|\alpha_{v'}|,\ |\beta_{v'}| < q_{v'}^{\frac{1}{2}}$. So the equality $\alpha_{v'} = q_{v'}^{\pm 1}\beta_{v'}$ does not hold. □

By Proposition 3.39,

$$\mathrm{Hom}_{o_{\mathscr{D}}}(\ker f_{R_{\mathscr{D}}}/(\ker f_{R_{\mathscr{D}}})^2,\ E_{\mathscr{D}}/o_{\mathscr{D}}) = \mathrm{Sel}_{\mathscr{D}}(F,\ \mathrm{ad}\,\rho)$$

and

$$\mathrm{Hom}_{o_{\mathscr{D}}}(\ker f_{R_{\mathscr{D}'}}/(\ker f_{R_{\mathscr{D}'}})^2,\ E_{\mathscr{D}}/o_{\mathscr{D}}) = \mathrm{Sel}_{\mathscr{D}'}(F,\ \mathrm{ad}\,\rho)$$

hold. Since we assume that $R_{\mathscr{D}} \simeq T_{\mathscr{D}}$, $\ker f_{R_{\mathscr{D}}}/(\ker f_{R_{\mathscr{D}}})^2 = \ker f/(\ker f)^2$, and the finiteness of $\mathrm{Sel}_{\mathscr{D}}(F,\ \mathrm{ad}\,\rho)$ follows since $T_{\mathscr{D}}$ is reduced. So the assumptions of level raising formalism (Theorem 2.8) is satisfied if we check

$$\mathrm{length}_{o_{\mathscr{D}}}\mathrm{Sel}_{\mathscr{D}'}(F,\ \mathrm{ad}\,\rho) \le \mathrm{length}_{o_{\mathscr{D}}}\mathrm{Sel}_{\mathscr{D}}(F,\ \mathrm{ad}\,\rho) + \mathrm{length}_{o_{\mathscr{D}}} o_{\mathscr{D}}/f(\Delta)o_{\mathscr{D}}$$

for the element Δ of $T_{\mathscr{D}}$ given in §10.

If $\mathrm{def}_{\mathscr{D}}(v) \neq$ **n.o.f**., by Lemma 3.37,

$$\mathrm{length}_{o_{\mathscr{D}}}\mathrm{Sel}_{\mathscr{D}'}(F,\ \mathrm{ad}\,\rho) \le \mathrm{length}_{o_{\mathscr{D}}}\mathrm{Sel}_{\mathscr{D}}(F,\ \mathrm{ad}\,\rho) + \mathrm{length}_{o_{\mathscr{D}}} H^0(F_v,\ \mathrm{ad}\,\rho(1) \otimes_{o_{\mathscr{D}}} E_{\mathscr{D}}/o_{\mathscr{D}}),$$

and it suffices to see

$$\mathrm{length}_{o_{\mathscr{D}}} H^0(F_v,\ \mathrm{ad}\,\rho(1) \otimes_{o_{\mathscr{D}}} E_{\mathscr{D}}/o_{\mathscr{D}}) = \mathrm{length}_{o_{\mathscr{D}}} o_{\mathscr{D}}/f(\Delta)o_{\mathscr{D}}.$$

This is a consequence of Proposition 10.5, 10.6, 10.8, and 10.9.

If $\mathrm{def}_{\mathscr{D}}(v) =$ **n.o.f**., one uses Lemma 3.36 and Proposition 10.3, and the inequality follows.

Thus we have shown that $(R_{\mathscr{D}'},\ T_{\mathscr{D}'}, \pi_{\mathscr{D}'},\ M_{\mathscr{D}'}, \langle\ ,\ \rangle_{\mathscr{D}'})$ is distinguished under the assumption that $(R_{\mathscr{D}},\ T_{\mathscr{D}}, \pi_{\mathscr{D}},\ M_{\mathscr{D}}, \langle\ ,\ \rangle_{\mathscr{D}})$ is distinguished by Theorem 2.8. Starting from the minimal case $\mathscr{D} = \mathscr{D}_{\min}$ (Theorem 9.1), we raise the level place by place by enlarging $\mathscr{D}$, and the general case is shown. □

Corollary 11.4. *Under the same assumption as in 11.1, the Selmer group* $\mathrm{Sel}_{\mathscr{D}}(F,\ \mathrm{ad}\,\rho)$ *of* $\rho = \rho_{\pi,E_\lambda}$ *for* π *appearing in* $T_{\mathscr{D}}$ *is finite.*

This is a consequence of the reducedness of $T_{\mathscr{D}}$.

Theorem 11.1 and 11.2 can be applied to the nearly ordinary Hecke algebra of Hida.

Let $\mathscr{D}$ be a deformation type of $\bar\rho$ as in Theorem 11.1. For $P = \mathrm{def}_{\mathscr{D}}^{-1}(\{\mathbf{n.o.}\})$, let $(o_{F_v}^\times)_\ell$ be the pro-ℓ completion of $o_{F_v}^\times$, $\mathscr{X}_{\mathbf{n.o.}}^{\mathrm{loc}} = \prod_{v\in P}(o_{F_v}^\times)_\ell$. We define the deformation type $\mathscr{D}^{\mathbf{n.o.}}$ in the following way.

- The deformation function of $\mathscr{D}^{\mathbf{n.o.}}$ is the same as that of $\mathscr{D}$.
- The coefficient ring $o_{\mathscr{D}^{n.o.}}$ of $\mathscr{D}^{\mathbf{n.o.}}$ is $o_{\mathscr{D}}[[\mathscr{X}_{\mathbf{n.o.}}^{\mathrm{loc}}]]$.
- At $v \in P^{\mathbf{n.o.}}$, the nearly ordinary type is

$$\kappa_{\mathscr{D},v}\cdot\mu_v : (I_{F_v}^{\mathrm{ab}})_{G_{F_v}} \simeq o_{F_v}^\times \longrightarrow o_{\mathscr{D}^{n.o.}}^\times.$$

  Here, $\kappa_{\mathscr{D},v}$ is the neary ordinary type of $\mathscr{D}$, and $\mu_v : o_{F_v}^\times \to (o_{\mathscr{D}}[[o_{F_v,\ell}^\times]])^\times \to o_{\mathscr{D}^{\mathbf{n.o.}}}^\times$ is the universal character. The flat twist type is the same as $\mathscr{D}$, by regarding it as a character with values in $o_{\mathscr{D}^{\mathbf{n.o.}}}^\times$.

Let $R_{\mathscr{D}^{\mathbf{n.o.}}}$ be the universal deformation ring of $\bar\rho$ of type $\mathscr{D}^{\mathbf{n.o.}}$. $R_{\mathscr{D}^{\mathbf{n.o.}}}$ is an $o_{\mathscr{D}^{\mathbf{n.o.}}}$-algebra. Since any deformation of type $\mathscr{D}$ is a deformation of type $\mathscr{D}^{\mathbf{n.o.}}$, there is a natural surjective map

$$f_{\mathscr{D}} : R_{\mathscr{D}^{\mathbf{n.o.}}} \longrightarrow R_{\mathscr{D}},$$

and $f_{\mathscr{D}}$ induces an isomorphism

$$R_{\mathscr{D}^{\mathbf{n.o.}}}/I^{\mathbf{n.o.}}R_{\mathscr{D}^{\mathbf{n.o.}}} \xrightarrow{\sim} R_{\mathscr{D}}.$$

Here, $I^{\mathbf{n.o.}}$ is the augumentation ideal of $o_{\mathscr{D}^{\mathbf{n.o.}}}$.

Corollary 11.5. *Assumptions are as in Theorem 11.1. Assume moreover that there is an* $R_{\mathscr{D}^{n.o.}}$*-module* $M_{\mathscr{D}^{n.o.}}$ *which has the following properties:*

1. $M_{\mathscr{D}^{\mathbf{n.o.}}}$ *is a free* $o_{\mathscr{D}^{\mathbf{n.o.}}}$*-module.*
2. $M_{\mathscr{D}^{\mathbf{n.o.}}}/I^{\mathbf{n.o.}}M_{\mathscr{D}^{\mathbf{n.o.}}} \xrightarrow{\sim} M_{\mathscr{D}}$ *as an* $R_{\mathscr{D}}$*-module.*
3. *Let* $T_{\mathscr{D}^{\mathbf{n.o.}}}$ *be the image of* $R_{\mathscr{D}^{\mathbf{n.o.}}}$ *in* $\mathrm{End}_{o_{\mathscr{D}^{\mathbf{n.o.}}}}(M_{\mathscr{D}^{\mathbf{n.o.}}})$*. Then* $T_{\mathscr{D}^{\mathbf{n.o.}}}$ *is reduced, and* $M_{\mathscr{D}^{\mathbf{n.o.}}}$ *is generically free of the same rank as the* $R_{\mathscr{D}}$*-rank of* $M_{\mathscr{D}}$ *(*$M_{\mathscr{D}}$ *is* $R_{\mathscr{D}}$*-free by Theorem 11.2).*

Under these assumptions, $R_{\mathscr{D}^{\mathbf{n.o.}}} \xrightarrow{\sim} T_{\mathscr{D}^{\mathbf{n.o.}}}$*,* $R_{\mathscr{D}^{\mathbf{n.o.}}}$ *is of relative complete intersection of dimension zero over* $o_{\mathscr{D}^{\mathbf{n.o.}}}$*, and* $M_{\mathscr{D}^{\mathbf{n.o.}}}$ *is a free* $R_{\mathscr{D}^{\mathbf{n.o.}}}$*-module.*

Proof of Corollary 11.5. Let α be the $R_{\mathscr{D}}$-rank of $M_{\mathscr{D}}$. By (2) and Nakayama's Lemma, there is a surjection $\beta : T_{\mathscr{D}^{\mathbf{n.o.}}}^{\oplus\alpha} \twoheadrightarrow M_{\mathscr{D}^{\mathbf{n.o.}}}$. By (3), β is an isomorphism generically, and hence is injective by the reducedness of $T_{\mathscr{D}^{\mathbf{n.o.}}}$. By (1), $T_{\mathscr{D}^{\mathbf{n.o.}}}$ is $o_{\mathscr{D}^{\mathbf{n.o.}}}$-free.

Let J be the kernel of $R_{\mathscr{D}^{\mathbf{n.o.}}} \twoheadrightarrow T_{\mathscr{D}^{\mathbf{n.o.}}}$. Since $T_{\mathscr{D}^{\mathbf{n.o.}}}$ is $o_{\mathscr{D}^{\mathbf{n.o.}}}$-free,

$$0 \longrightarrow J/I^{\mathbf{n.o.}}J \longrightarrow R_{\mathscr{D}^{\mathbf{n.o.}}}/I^{\mathbf{n.o.}}R_{\mathscr{D}^{\mathbf{n.o.}}} \xrightarrow{\gamma} T_{\mathscr{D}^{\mathbf{n.o.}}}/I^{\mathbf{n.o.}}T_{\mathscr{D}^{\mathbf{n.o.}}} \longrightarrow 0$$

is exact. γ is an isomorphism by Theorem 11.1. Thus $J = 0$ by Nakayama's lemma, and $R_{\mathscr{D}^{\mathbf{n.o.}}} \xrightarrow{\sim} T_{\mathscr{D}^{\mathbf{n.o.}}}$ holds. Since $R_{\mathscr{D}^{\mathbf{n.o.}}}$ is $o_{\mathscr{D}^{\mathbf{n.o.}}}$-finite flat and $R_{\mathscr{D}^{\mathbf{n.o.}}}/I^{\mathbf{n.o.}}R_{\mathscr{D}^{\mathbf{n.o.}}}$ is of relative complete intersection of dimension zero over $o_{\mathscr{D}^{\mathbf{n.o.}}}/I^{\mathbf{n.o.}} = o_{\mathscr{D}}$, $R_{\mathscr{D}^{\mathbf{n.o.}}}$ is of relative complete intersection of dimension zero over $o_{\mathscr{D}^{\mathbf{n.o.}}}$. □

Using the results in §5 and the perfect complex argument, the existence of $T_{\mathscr{D}^{\mathbf{n.o.}}}$ and $M_{\mathscr{D}^{\mathbf{n.o.}}}$ as in Corollary 11.5 is shown (exact control theorem). If $\mathrm{def}_{\mathscr{D}}(v) = \mathbf{n.o.}$ for any $v|\ell$, $T_{\mathscr{D}^{\mathbf{n.o.}}}$ is the nearly ordinary Hecke algebra of Hida.

References

[1] Bloch, S., Kato, K.: The Grothendieck Festschrift, Vol. I, 333–400, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[2] Bourbaki, N.: Algèbre Commutative, chap.3–4, Hermann 1961.

[3] Blasius, D., Rogawski, J.D.: Motives for Hilbert modular forms. Invent. Math. 114 (1993), no. 1, 55–87.

[4] Conrad, B.: The flat deformation functor. Modular forms and Fermat's last theorem (Boston, MA, 1995), 373–420, Springer, New York, 1997.

[5] Carayol, H.: Sur la mauvaise réduction des courbes de Shimura, Compositio Math. 59 (1986), no. 2, 151–230.

[6] Carayol, H.: Sur les représentations p-adiques associées aux formes modulaires de Hilbert, Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 3, 409–468.

[7] Cline, E., Parshall, B., Scott, L.: Cohomology of finite groups of Lie type. I. Inst. Hautes Études Sci. Publ. Math. No. 45 (1975), 169–191.

[8] de Jong, A.J.: Smoothness, semi-stability and alterations. Inst. Hautes Études Sci. Publ. Math. No. 83 (1996), 51–93.

[9] Diamond, F.: On deformation rings and Hecke rings. Ann. of Math. (2) 144 (1996), no. 1, 137–166.

[10] Diamond, F.: The Taylor-Wiles construction and multiplicity one. Invent. Math. 128 (1997), no. 2, 379–391.

[11] Deligne, P.: Travaux de Shimura Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, pp. 123–165. Lecture Notes in Math., Vol. 244, Springer, Berlin, 1971.

[12] Diamond, F., Taylor, R.: Lifting modular mod ℓ representations. Duke Math. J. 74 (1994), no. 2, 253–269.

[13] Diamond, F., Taylor, R.: Nonoptimal levels of mod ℓ modular representations. Invent. Math. 115 (1994), no. 3, 435–462.

[14] Fontaine, J.M., Lafaille, G.: Constructions de représentations p-adiques Ann. Sci. École Norm. Sup. (4) 15 (1982), no. 4, 547–608.

[15] Fujiwara, K.: Level optimization in the totally real case, preprint, arXiv.math.NT/0602586.

[16] Fujiwara, K.: Galois deformations and arithmetic geometry of Shimura varieties, Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006, European Mathematical Society, to appear.

[17] Grothendieck, A., Dieudonné, J.: Éléments de géométrie algébrique IV, Première Partie, Publications Mathématiques, 20, 1964.

[18] Grothendieck, A., Dieudonné, J.: Éléments de géométrie algébrique IV, Seconde Partie, Publications Mathématiques, 24, 1965.

[19] Géradin, P.: Facteurs locaux des algebres simples de rang 4, I , Groupes réductif et formes automorphes, I, pp. 37–77, Publ. Math. Univ. Paris VII, 1, Univ. Paris VII, Paris, 1978.

[20] Grothendieck, A.: Le groupe de Brauer. III. Exemples et compléments, 1968 Dix Exposés sur la Cohomologie des Schémas pp. 88–188 North-Holland, Amsterdam; Masson, Paris.

[21] Grothendieck, A.: Modèles de Néron et monodromie Groupes de Monodromie en Géométrie Algébrique, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I). Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. Lecture Notes in Mathematics, Vol. 288. Springer-Verlag, Berlin-New York, 1972.

[22] Hida, H.: Galois representations into $\mathrm{GL}_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms. Invent. Math. 85 (1986), no. 3, 545–613.

[23] Hida, H.: On p-adic Hecke algebras for GL_2 over totally real fields. Ann. of Math. (2) 128 (1988), no. 2, 295–384.

[24] Hida, H.: On nearly ordinary Hecke algebras for GL(2) over totally real fields. Algebraic number theory, 139–169, Adv. Stud. Pure Math., 17, Academic Press, Boston, MA, 1989.

[25] Jarvis, F.: Mazur's principle for totally real fields of odd degree, Compositio Math. 116 (1999), no. 1, 39–79.

[26] Jarvis, F.: Level lowering for modular mod l representations over totally real fields. Math. Ann. 313 (1999), no. 1, 141–160.

[27] Jacquet, H., Langlands, R.P.: Automorphic forms on GL(2). Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970.

[28] Jacquet, H., Shalika, J.A.: On Euler products and the classification of automorphic forms. II. Amer. J. Math. 103 (1981), no. 4, 777–815.

[29] Kutzko, P.: The Langlands conjecture for GL(2) of a local field. Ann. of Math. (2) 112 (1980), no. 2, 381–412.

[30] Langlands, R.P.: Base change for GL(2). Annals of Mathematics Studies, 96. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980.

[31] Lenstra, H. W.: Complete intersections and Gorenstein rings. Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), 99–109, Ser. Number Theory, I, Internat. Press, Cambridge, MA, 1995.

[32] Mazur, B.: Deforming Galois representations. Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 385–437, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989.

[33] Ohta, M.: On ℓ-adic representations attached to automorphic forms. Japan. J. Math. (N.S.) 8 (1982), no. 1, 1–47.

[34] Ohta, M.: On the zeta function of an abelian scheme over the Shimura curve. Japan. J. Math. (N.S.) 9 (1983), no. 1, 1–25.

[35] Rajaei, A.: On the levels of mod ℓ Hilbert modular forms. J. Reine Angew. Math. 537 (2001), 33–65.

[36] Raynaud, M.: Schémas en groupes de type $(p, \ldots, p)$. Bull. Soc. Math. France 102 (1974), 241–280.

[37] Ramakrishna, R.: On a variation of Mazur's deformation functor. Compositio Math. 87 (1993), no. 3, 269-286.

[38] Ribet, K.A.: Congruence relations between modular forms. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 503–514, PWN, Warsaw, 1984.

[39] Ribet, K.A.: On modular representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms. Invent. Math. 100 (1990), no. 2, 431–476.

[40] Shimura, G.: On canonical models of arithmetic quotients of bounded symmetric domains. Ann. of Math. (2) 91 1970 144–222.

[41] Saito, T.: Modular forms and p-adic Hodge theory. Invent. Math. 129 (1997), no. 3, 607–620.

[42] Saito, T.: Hilbert modular forms and p-adic Hodge theory. Preprint (1999).

[43] Schlessinger, M.: Functors of Artin rings. Trans. Amer. Math. Soc. 130, 1968, 208–222.

[44] Shimizu, H.: Theta series and automorphic forms on GL(2). J. Math. Soc. Japan 24 (1972), 638–683.

[45] Tate, J.: p-divisible groups. 1967 Proc. Conf. Local Fields (Driebergen, 1966) pp. 158–183 Springer, Berlin.

[46] Taylor, R.: On Galois representations associated to Hilbert modular forms. Invent. Math. 98 (1989), no. 2, 265–280.

[47] Taylor, R.: On Galois representations associated to Hilbert modular forms. II. Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), 185–191, Ser. Number Theory, I, Internat. Press, Cambridge, MA, 1995.

[48] Tsuji, T.: p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case. Invent. Math. 137 (1999), no. 2, 233-411.

[49] Taylor, R, Wiles, A.: Ring-theoretic properties of certain Hecke algebras. Ann. of Math. (2) 141 (1995), no. 3, 553–572.

[50] Wiles, A.: On ordinary Λ-adic representations associated to modular forms. Invent. Math. 94 (1988), no. 3, 529–573.

[51] Wiles, A.: Modular elliptic curves and Fermat's last theorem. Ann. of Math. (2) 141 (1995), no. 3, 443–551.